User's manual for DISODE45, a MATLAB integration code (v1.0)

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1 The DISODE45 function

DISODE45 is a Matlab function that solves non-smooth, non-stiff differential systems

$$y'(t) = f(t, y(t)), \qquad y(0) = y_0 \in \mathbb{R}^m, \qquad t \in [t_0, t_f]$$

The non-smoothness of the solution $y = y(t)$ happens at points, switching points, (t_d, y_d) where one of several switching surfaces defined by a function $g(t, y) = (g_1(t, y), \dots, g_k(t, y))$ vanish, that is, at least for a index i, $g_i(t_d, y_d) = 0$.

The call to this function has the following syntax

```
[T,Y] = disode45(odefun,switchfun,tspan,y0)
[T,Y,TDIS,YDIS,IDIS,STATS]=disode45(odefun,switchfun,tspan,y0)
[T,Y] = disode45(odefun,switchfun,tspan,y0,options)
[T,Y,TDIS,YDIS,IDIS,STATS]=disode45(odefun,switchfun,tspan,y0,options)
```

It is very similar to the syntax used by the ODE suite Matlab package so that users that are familiar with this software can find the use of `disode45` very easy.

1.1 Input arguments

odefun A function handle that evaluates the right side of the differential equations $f(t, y)$.

switchfun A function handle that evaluates the function $g(t, y)$ defining the switching surfaces.

tspan A vector specifying the interval of integration, $[t0, tf]$. The solver imposes the initial conditions at `tspan(1)`, and integrates from `tspan(1)` to `tspan(end)`. For the moment, the code does not allow vectors `tspan` with more than two components, and t_f must be greater than t_0

y0 A m dimensional vector of initial conditions.

options Structure of optional parameters that change the default integration properties. You can create options using the disodeset function.

1.2 Output arguments

T Column vector of time points.

Y Solution array. Each row in Y corresponds to the solution at a time returned in the corresponding row of T.

TDIS Vector of times at which discontinuities or switching points of the numerical solution have been detected.

YDIS The solution at the times of switching points. Each row in YDIS corresponds to the solution at a time returned in the corresponding row of TDIS.

IDIS Vector containing the indexes i of the switching function that vanishes at each switching point.

- The absolute value of `IDIS(i)` indicates the switching function corresponding to the i–eme switching point.
- The sign of `IDIS(i)` indicates the type of discontinuity. It it is positive, the discontinuity is transversal. If it is negative, the switching point is Filippov (entering or either exiting a sliding region). For example, a value `IDIS(4)` $= -2$ means that the fourth switching point is a Filippov point located at the second switching surface $g_2(t, y)$.

STATS Vector containing some statistics about the integration

STATS(1)	Number of accepted normal steps
STATS(2)	Number of rejected normal steps
STATS(3)	Number of accepted sliding steps
STATS(4)	Number of rejected sliding steps
STATS(5)	Number of steps at with a possible switching point
STATS(6)	Number of transversal discontinuities
STATS(7)	Number of sliding discontinuities
STATS(8)	Number of exits of a sliding region
STATS(9)	Number of calls to odefun
STATS(10)	Number of calls to switchfun
STATS(11)	Number of calls to gradswitchfun

1.3 Required user's functions

odefun The function,

```
f=odefun(t,y)
```

for a scalar t and a column vector y, must return a column vector f corresponding to $f(t, y)$.

switchfun The function

```
[value,isterminal,direction] = switchfun(t,y)
```

for a scalar t and a column vector y, must return three column vectors

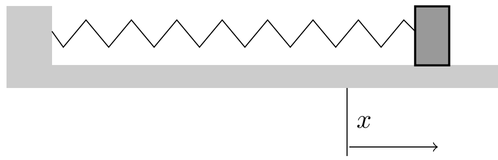
- `value(i)` is the value of the i-th component $g_i(t, y)$
- `isterminal(i)` specifies the action to be taken when a zero of the function $g_i(t, y)$ is found. It can take the values -1, 0 or 1.
  - `isterminal(i)`=1 means that the integration must terminate at a zero of the i–th switching function.
  - `isterminal(i)`=0 means that the integration must continue without any action.
  - `isterminal(i)`=-1 means that the program must call an external function `actionatswitch(t,y)`, that should have been provided by the user in the options by means of disodeset function (see sections 3 and 4).
- `direction(i)` specifies which zeros of $g_i(t, y)$ have to be computed.
  - `direction(i) = 0` means that all zeros are to be computed (the default).
  - `direction(i) = +1` means that only the zeros where the switching function increases, that is, it passes from negative to positive, must be computed.
  - `direction(i) = -1` means that only the zeros where the switching function decreases must be computed.

1.4 Getting the code

The code can be downloaded at http://iuma.unizar.es/en/research/software

2 A simple first example

Let us consider a simple mechanical system consisting on a mass $m = 1$ atached to a fixed wall by a spring with stiffness coefficient 1, moving on a surface with a Coulomb friction with friction coefficient μ so that the friction force is $F_C = \mu g = 0.4$.



This system is modelled by the non-smooth second order differential equation

$$x'' = -x - F_C \text{ sign}(x').$$

In order to be integrated by `DISODE45`, this second order equation must be expressed as a first order system with two components $y_1(t) = x(t)$ and $y_2(t) = x'(t)$, as

$$y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ -y_1 - F_C \text{ sign}(y_2) \end{pmatrix} = f(t, y)$$

Clearly the vector field is non smooth at points where $x' = y_2 = 0$. Then, we have a unique switching surface $g(y_1, y_2) = y_2$.

The solution, with initial conditions $x(0) = 3, x'(0) = 0$, passes first through 3 transversal discontinuities and after that, at $t = 12.5664\ldots$ it enters a sliding region with $x = 0.2$, $x' = 0$ and stays at this point forever (the friction force is greater than the force of the spring and the mass stops).

The evolution of the solution $x(t)$ and its derivative $x'(t)$ are depicted in Figure 1. The discontinuity points are indicated by means of small circles. Note that for this problem, since it is a second order differential equation, the solution $x(t)$ and its derivative $x'(t)$ are continuous, as it can be seen in the plots, but the second derivative $x''(t)$ is not continuous.

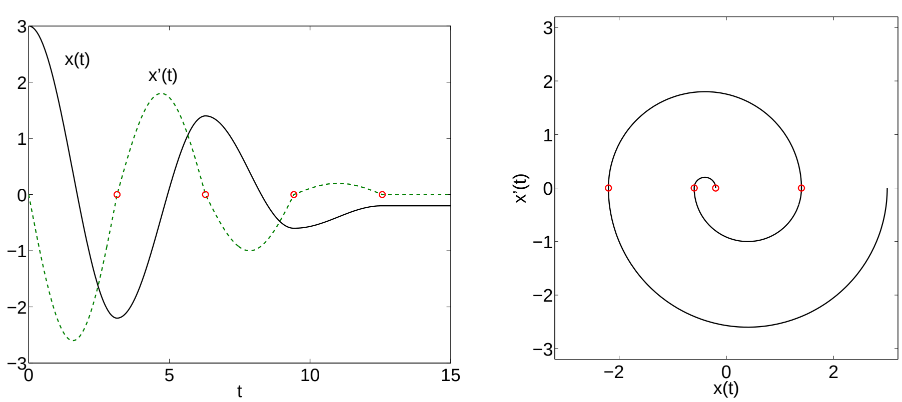


Figure 1: Left: Solution and derivative against time, Right: Phase diagram

A Matlab code to integrate this problem with `disode45` can be

```
%
%  Call to disode45
%
 y0=[3;0];
 [tout,yout,tdis,ydis,idis,stats]=disode45(@fun, @gfun,[0,20], y0);
%
%  Definition of the vector field
%
 function f=fun(t,y)
      Fc=0.4;
      f=[y(2);-y(1)-Fc*sign(y(2))] ;
 end
%
%  Definition of the switching surface
%
 function [g,isterminal,direction]=gfun(t,y)
      g=y(2);
      isterminal=0;
      direction=0;
 end
```

Once the integration of this problem has concluded, the vectors `tdis`, `ydis` and `idis` contain the following data

```
tdis =

     0    3.1416    6.2831    9.4247   12.5661

ydis =

    3.0000         0
   -2.2000   -0.0000
    1.3999   -0.0000
   -0.5999    0.0000
   -0.2001   -0.0000

idis =

     1     1     1     1    -1
```

The elements of the vector `idis` indicate that all the switching points correspond to the first switching surface, an expected fact because there is a unique switching surface. The first four components are positive, which means that these discontinuities are transversal. The last one is negative, and therefore this discontinuity starts a sliding region. Note that the integration starts at an initial condition $(3, 0)$ which belongs to the switching surface. This is why the vector `tdis` has 0 as its first element, which means that there is a switching point at $t = 0$.

The plots shown in Figure 1 (without legends) can be obtained with the matlab orders

```
plot(tout,yout(:,1),'k',tout,yout(:,2),'k--',tdis,ydis(:,2),'ro')
```

```
plot(yout(:,1),yout(:,2),'k-',ydis(:,1),ydis(:,2),'ro')
```

3 The DISODESET function

`DISODESET` is a Matlab function that creates a options structure that lets the user set some parameters (options) to be used by `DISODE45` in the numerical integration.

The call to this function has the following syntax

```
options=disodeset('name1',value1,'name2',value2,...)
```

3.1 Allowed options

'RelTol' Sets the Relative error tolerance (default to 1e-3). Each integration step satisfies $||est||_2 \leq RelTol * ||y||_2 + AbsTol$.

'AbsTol' Sets the absolute error tolerance (default to 1e-6). See RelTol.

'InitialStep' Sets a suggested initial step size and must be a positive scalar. The solver will try this first. By default the solver determine an initial step size automatically.

'EventControl' Sets the type of control for the detection of discontinuities

- 0 Existence of discontinuity is checked at every step and every stage of failed steps.
- 1 Existence of discontinuity is checked at every stage of every step.
- k Existence of discontinuity is checked at every stage of every step and at 6*k uniformly distributed points inside every step.

'Refine' Sets the output refinement factor and must be a positive integer. This property increases the number of output points by the specified factor producing smoother output. Refine defaults to 4.

'Gradient' Specifies, by means of a function handle, the function to compute the gradient of the switching functions.

'ActionSwitch' Specifies, by means of a function handle, the function to be called by the integrator when a switching point is found. This output function is called if the corresponding value of the vector `isterminal` is -1. ActionSwitch defaults to [ ].

4 Optional functions that can be provided by the user

gradswitchfun The function

```
grad = gradswitchfun(t,y,inddis)
```

for a scalar `t`, column vector `y`, and a switching function index `inddis`, must return a column vector grad which is the gradient vector of the function g_{inddis}

This function must be provided by the user if the option 'Gradient' is set. By default, disode45 computes the gradient vector by means of divided differences.

actionatswitch The function

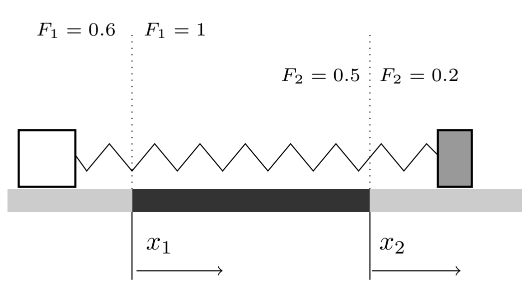
```
yout = actionatswitch(t,y)
```

for a scalar `t` and a column vector `y`, must return a column vector `yout` which is the new state vector. The integration will proceed from the point t with the initial condition `yout`.

This function must be provided if the option 'ActionSwitch' is set, and it is executed if the function `switchfun(t,y)` returns a value `-1` in the vector `isterminal` at the switching point.

5 Examples

Example 1 In the first example we consider two masses $m_1 = m_2 = 1$ linked by a spring, moving on a surface with Coulomb friction (see [1, pp. 162]). Both masses are equal, but made with different material, so their friction coefficients μ_1 and μ_2 are different and so are the friction forces F_1 and F_2. The surface where the masses move has two physically different parts, which makes the friction coefficients change.



This mechanical system is modelled by the non-smooth differential system

$$\begin{cases} x_1'' = -(x_1 - x_2) - F_1 \mathrm{sign}(x_1'), \\ x_2'' = -(x_2 - x_1) - F_2 \mathrm{sign}(x_2'), \\ F_1 = \begin{cases} 1 & \text{if } x_1 \leq 0 \\ 0.6 & \text{if } x_1 > 0 \end{cases} \qquad F_2 = \begin{cases} 0.5 & \text{if } x_2 \leq 0 \\ 0.2 & \text{if } x_2 > 0 \end{cases} \\ x_1(0) = -2,\ x_2(0) = 3,\ x_1'(0) = 0,\ x_2'(0) = 0 \\ t \in [0, 6]. \end{cases}$$

Again, we must express the second order system as a first order system with four components $y_1(t) = x_1(t)$, $y_2(t) = x_2(t)$, $y_3(t) = x_1'(t)$ and $y_4(t) = x_2'(t)$)

$$y' = \begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{pmatrix} = \begin{pmatrix} y_3 \\ y_4 \\ -(y_1 - y_2) - F_1 \ \mathrm{sign}(y_3) \\ -(y_2 - y_1) - F_2 \ \mathrm{sign}(y_4) \end{pmatrix} = f(t, y)$$

Clearly the vector field $f(t, y)$ is non smooth at points where $x_1' = y_3 = 0$, $x_2' = y_4 = 0$, $x_1 = y_1 = 0$ or $x_2 = y_2 = 0$. Then, we have four switching surfaces $g_1(y) = y_1$, $g_2(y) = y_2$, $g_3(y) = y_3$ and $g_4(y) = y_4$.

The solution, with these initial conditions, passes first through 14 transversal discontinuities and after some time it enters a sliding region onto $g_1(y) = 0$. Inside this sliding region, $y_1 = 0$, the solution crosses three times transversal discontinuities and finally it attains a co-dimension 2 sliding region $y_3 = y_4 = 0$, where both masses do not move, and the system remains there forever. The integrator stops when it find a co-dimension 2 Filippov point.

The components of the solution $x_1(t)$ and $x_2(t)$ of this problem as well as their derivatives are depicted in Figure 2. The 18 discontinuity points are indicated by means of small circles.

A Matlab code to integrate this problem with `disode45` can be

```
%
%  Call to disode45
%
 y0 = [-2;3;0;0];
 [tout,yout,tdis,ydis,idis,stats]=disode45(@fun, @gfun,[0,20], y0);
%
%  Definition of the vector field
%
 function f=fun1(t,y)
```

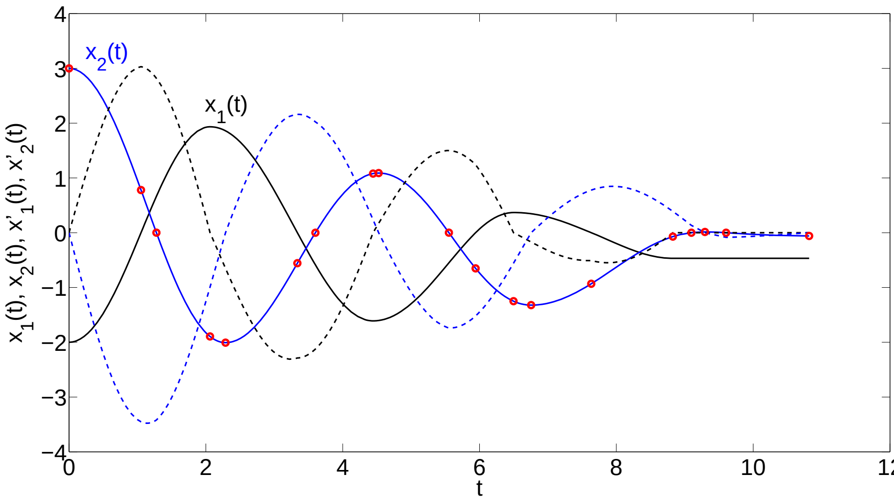


Figure 2: Solution for Example 1

```
   E=1.0;
   if y(1) > 0
      mu1=1.0;
   else
      mu1=0.6;
   end
   if y(2) > 0
      mu2=0.2;
   else
      mu2=0.5;
   end
   f=[y(3);y(4);-E*(y(1)-y(2))-mu1*sign(y(3)); ...
                    -E*(y(2)-y(1))-mu2*sign(y(4))] ;
 end
%
%  Definition of the switching surfaces
%
 function [g,isterminal,direction]=gfun1(t,y)
   g=[y(3); y(4); y(1); y(2)];
   isterminal=[0;0;0;0];
   direction=[0;0;0;0];
 end
```

Once the integration of this problem has concluded, the vectors `tdis`, `ydis` contain the switching point and `idis` contain the following data

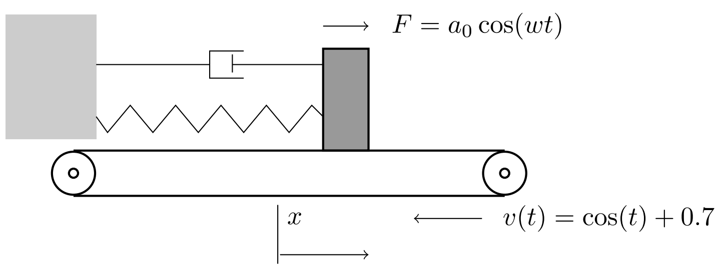
```
idis =

     1     3     4     1     2     3     4     1
     2     4     3     1     2     3    -1     4
     2     4    -2
```

The elements of the vector `idis` indicate that the first switching point correspond to the first switching surface, the second one to the third surface and so on. The positive elements mean that these discontinuities are transversal. The negative ones mean that the solution enters into a sliding region.

The plots in Figure 2 (without legends) can be obtained with the matlab order

```
plot(tout,yout(:,1),'k',tout,yout(:,2),'b', tout, ...
   yout(:,3),'k--',tout,yout(:,4),'b--',tdis,ydis(:,2),'ro');
```

Example 2 In the second example (see [4]) we consider a mass $m = 1$ linked to a wall by a spring of stiffness k and a damper of viscous damping coefficient r. A external force $F = a_0 \cos(wt)$ is acting upon the mass which is placed onto a belt that moves with velocity $v(t) = \cos(t) + 0.7$. Therefore, a Coulomb friction force F_C acts upon the mass.



This mechanical system is modelled by the non-smooth second order differential system

$$\begin{cases} x'' = -kx - 2rx' + a_0 \cos(wt) - F_C \operatorname{sign}(x' - v(t)), \\ F_C = 0.4 \qquad a_0 = 1, \quad r = 0.2, \quad k = 1, \quad w = 0.7, \\ v(t) = \cos(t) + 0.7, \\ \\ x(0) = 3, x'(0) = 0, \\ t \in [0, 30]. \end{cases}$$

Expressed as a first order system with two components $y_1(t) = x(t)$, $y_2(t) = x'(t)$, we have

$$y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ -ky_1 - 2ry_2 + a_0 \cos(wt) - F_C \operatorname{sign}(y_2 - v(t)) \end{pmatrix} = f(t, y)$$

Clearly the vector field $f(t, y)$ is non smooth at points where $x'(t) = y_2(t) = v(t)$. Then, we have one switching surface $g(y) = y_2 - v(t)$, that depends on y_2 and also on $v(t)$.

The solution $x(t)$ and its derivative $x'(t)$ are depicted in the upper plot of Figure 3. The discontinuity points (where the second derivative $x''(t)$ is not continuous) are indicated by means of small circles. To visualize the sliding regions, we give in the down plot of Figure 3 the function $x'(t) - v(t)$. The sliding regions correspond to the intervals at which the functions vanishes.

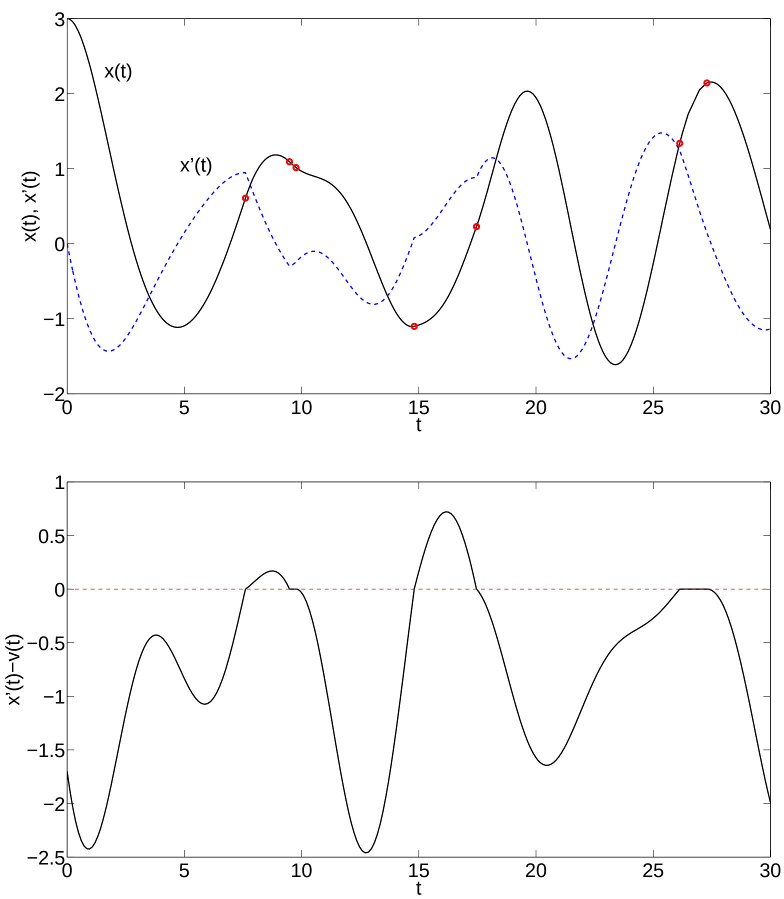


Figure 3: Example 2, Solution (top) and Sliding regions (dowm)

The solution, with the considered initial conditions, passes first through a transversal discontinuity, then it enters a sliding region for a short time until it exits it, then it passes through two transversal discontinuities and then it enters into another sliding region.

A Matlab code to integrate this problem with `disode45` can be

```
%
%  Call to disode45
%
 y0 = [3;0];
 [tout,yout,tdis,ydis,idis,stats]=disode45(@fun, ...
                                    @gfun,[0,30], y0);
%
%  Definition of the vector field
%
 function f=fun(t,y)
      k=1.0;
      r=0.2;
      a0=1.0;
      w=0.7;
      Fc=0.4;
      v=cos(t)+0.7;
      f=[y(2);-k*y(1)-2*r*y(2)+a0*cos(w*t)-Fc*sign(y(2)-v)];
 end
%
%  Definition of the switching surfaces
%
 function [g,isterminal,direction]=gfun(t,y)
    v=cos(t)+0.7;
    g=y(2)-v;
    isterminal=0;
    direction=0;
 end
```

Once the integration of this problem concludes, the vectors tdis, ydis and idis contain the following data

```
tdis =

    7.6056    9.4835    9.7653   14.8051   17.4589   26.1318   27.2913

ydis =

    0.6072    0.9459
    1.0923   -0.2983
    1.0143   -0.2426
   -1.1013    0.0806
    0.2269    0.8792
    1.3390    1.2411
```

```
    2.1418    0.1455

idis =

     1    -1    -1     1     1    -1    -1
```

In this case, the second element of `idis` equal to -1 means that the solution enters into a sliding region, and the third one, also -1, means that the solution exits from the sliding region. The same happens with the last two elements.

The plots in Figure 3 (without legends) can be obtained with the matlab orders

```
plot(tout,yout(:,1),'k',tout,yout(:,2),'k--', tdis, ...
                                          ydis(:,1),'ro');
plot(tout,yout(:,2)-0.7-cos(tout)',[0,30],[0,0],'r');
```

Example 3 In the third example (see A. Luo [4, pp. 115]) we consider two masses m_1 and m_2 linked to a wall by springs of stiffness k_1, k_2 respectively and dampers of viscous damping coefficients r_1, r_2. A external force $F = a_0 \cos(wt)$ is acting upon the mass m_1. Mass m_1 is placed into a hole in mass m_2 so that the masses can hit each other as depicted in Figure 4.

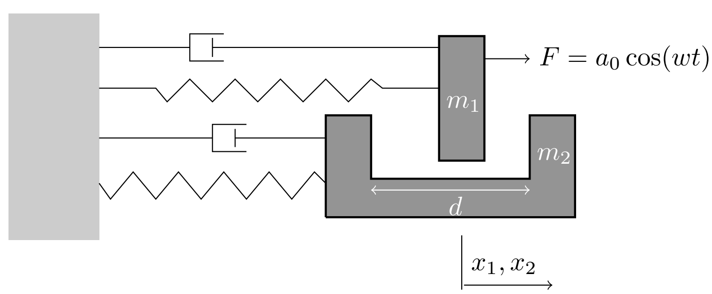


Figure 4: Example 3

Whenever both masses have no contact, that is, $y_1 - y_2 > d/2$ or $y_2 - y_1 > d/2$, this mechanical system is modelled by the differential system

$$\begin{aligned} m_1 x_1'' &= -k_1 x_1 - r_1 x_2' + a_0 \cos(wt) + b_0 \\ m_2 x_2'' &= -k_2 x_2 - r_2 x_2' \end{aligned} \tag{1}$$

When the masses hit each other, $|x_1 - x_2| = d/2$, the behaviour of the system depends on their velocity. If they are different, $x_1' \neq x_2'$, we have a switching point in which the vector field does not change but the initial conditions of the system change from (x_1, x_2, x_1', x_2') to $(\hat{x}_1, \hat{x}_2, \hat{x}_1', \hat{x}_2')$

according to the following formulas

$$\begin{aligned}
\hat{x}_1 &= x_1, \\
\hat{x}_2 &= x_2, \\
\hat{x}_1' &= \frac{m_1 - m_2 e}{m_1 + m_2}\, x_1' + \frac{(1+e)m_2}{m_1 + m_2}\, x_2', \\
\hat{x}_2' &= \frac{m_2 - m_1 e}{m_1 + m_2}\, x_1' + \frac{(1+e)m_1}{m_1 + m_2}\, x_2'.
\end{aligned}$$

If the masses hit with the same velocity, the system enters into a sticking region that is governed by the differential system

$$\begin{aligned}
(m_1 + m_2)x_1'' &= -(k_1 + k_2)x_1 - (r_1 + r_2)x_1') + \frac{k_2 d}{2} + a_0 \cos(wt) + b_0, \\
(m_1 + m_2)x_2'' &= -(k_1 + k_2)x_2 - (r_1 + r_2)x_2') - \frac{k_1 d}{2} + a_0 \cos(wt) + b_0,
\end{aligned}$$

if $x_1 = x_2 + d/2$ or

$$\begin{aligned}
(m_1 + m_2)x_1'' &= -(k_1 + k_2)x_1 - (r_1 + r_2)x_1') - \frac{k_2 d}{2} + a_0 \cos(wt) + b_0, \\
(m_1 + m_2)x_2'' &= -(k_1 + k_2)x_2 - (r_1 + r_2)x_2') + \frac{k_1 d}{2} + a_0 \cos(wt) + b_0,
\end{aligned}$$

if $x_2 = x_1 + d/2$.

Note that in this region, since $|x_1 - x_2| = d/2$, it holds $x_1''(t) = x_2''(t)$ and $x_1'(t) = x_2(t)$. The system stays in this region while the forces per unit mass, that act upon each mass

$$\begin{aligned}
u_1 &= \frac{F_1}{m_1} = -\frac{r1}{m1}x_1' - \frac{k1}{m1}x_1 + \frac{b_0 + a_0 \cos(wt)}{m1} \\
u_2 &= \frac{F_2}{m_2} = -\frac{r2}{m2}x_2' - \frac{k2}{m2}x_2
\end{aligned}$$

satisfy $x_1 - x_2 = d/2$ and $u_1 \geq u_2$ or well $x_2 - x_1 = d/2$ and $u_2 \geq u_1$. In other case, the systems goes back to the non sticking evolution and equations (1) govern the evolution of the system.

With the above considerations, and defining the state vector by $y(t) = (y_1(t), y_2(t), y_3(t), y_4(t))^T = (x_1(t), x_2(t), x_1'(t), x_2'(t))^T$ the system can be modelled by the non smooth system

$$y'(t) = A(t)y(t) + b(t),$$

with

$$A(t) = \begin{cases} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_1+k_2}{m_1+m_2} & -\dfrac{r_1+r_2}{m_1+m_2} & 0 & 0 \\ -\dfrac{k_1+k_2}{m_1+m_2} & -\dfrac{r_1+r_2}{m_1+m_2} & 0 & 0 \end{pmatrix}, & \begin{array}{l} \text{if } y_1 - y_2 \geq d/2 \text{ and } u_1 \geq u_2 \\ \text{or } y_2 - y_1 \geq d/2 \text{ and } u_2 \geq u_1 \end{array} \\ \\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_1}{m_1} & -\dfrac{r_1}{m_1} & 0 & 0 \\ -\dfrac{k_2}{m_2} & -\dfrac{r_2}{m_2} & 0 & 0 \end{pmatrix}, & \text{otherwhise.} \end{cases}$$

$$b(t) = \begin{cases} \begin{pmatrix} 0 \\ 0 \\ \dfrac{k_2 d/2}{m_1+m_2} + \dfrac{a_0 \cos(wt)}{m_1+m_2} \\ \dfrac{-k_1 d/2}{m_1+m_2} + \dfrac{a_0 \cos(wt)}{m_1+m_2} \end{pmatrix}, & \text{if } y_1 - y_2 \geq d/2 \text{ and } u_1 \geq u_2 \\ \\ \begin{pmatrix} 0 \\ 0 \\ \dfrac{-k_2 d/2}{m_1+m_2} + \dfrac{a_0 \cos(wt)}{m_1+m_2} \\ \dfrac{k_1 d/2}{m_1+m_2} + \dfrac{a_0 \cos(wt)}{m_1+m_2} \end{pmatrix}, & \text{if } y_2 - y_1 \geq d/2 \text{ and } u_2 \geq u_1 \\ \\ \begin{pmatrix} 0 \\ 0 \\ \dfrac{b_0}{m1} + \dfrac{a_0}{m1}\cos(wt) \\ 0 \end{pmatrix}, & \text{otherwhise} \end{cases}$$

In this example we have taken

$r_1 = 0.6, \quad r_2 = 0.6, \quad k_1 = 30, \quad k_2 = 20, \quad a_0 = 1, \quad w = 0.7, \quad b_0 = 35$

and as initial conditions

$$y_1(0) = 0.2, y_2(0) = 0.3, y_1'(0) = 0, y_2'0) = 0, \quad t \in [0, 10].$$

The vector field $f(t, y)$ is non smooth at points where $y_1 - y_2 = d/2$, when $y_2 - y_1 = d/2$. Starting with the selected initial conditions, the system evolves smoothly until the masses hit. Then, a series of hits happen repeatedly, each time after less time. After a finite time, but infinite hits, the masses hit with the same velocity and the system enters into a sticking region. After some time, the system leaves this region and the behaviour repeats the same process. In the bottom plot of Figure 5 we depict the function $x_1(t) - x_2(t)$. The points where $x_1(t) - x_2(t) = 0.5$ correspond to points where the masses hit, and the intervals where $x_1(t) - x_2(t)$ correspond to sticking regions. In the top plot of Figure 5 we show the solution $x_1(t), x_2(t)$ against time t. again, the discontinuity points are indicated by means of small circles.

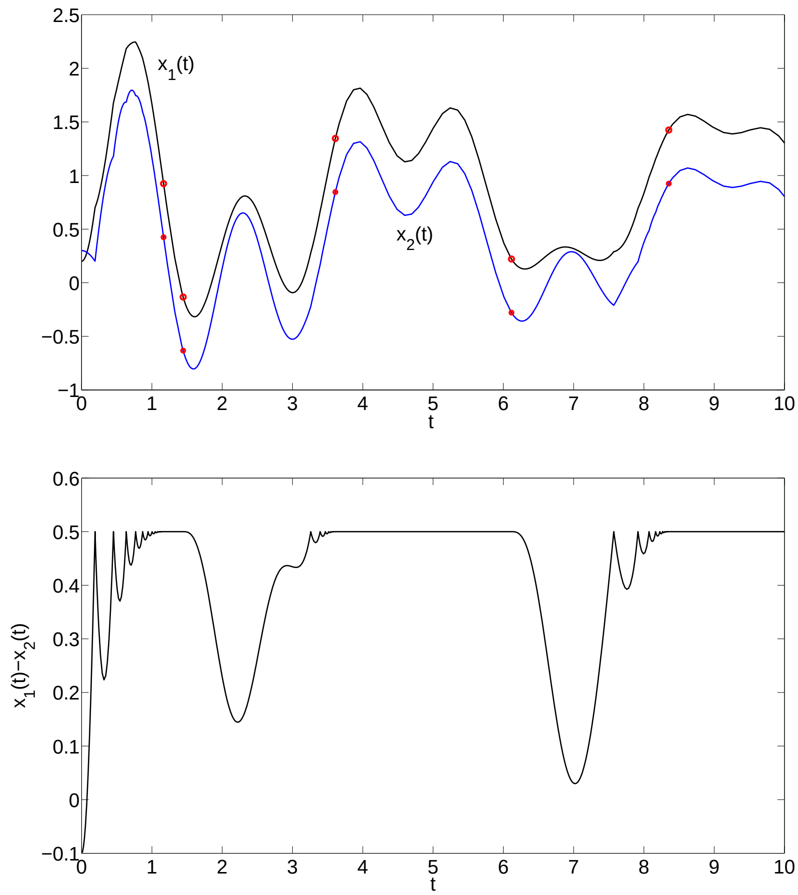


Figure 5: Example 3, Solution (top) and Sliding regions (bottom)

This problem has two specially complicated characteristics. On one side, the initial point of a sticking region is in fact a accumulation point of discontinuities, and in theory to attain it we should detect an infinity number of switching points. To solve this, the start of the sticking region can be assumed when $|x_1 - x_2| = d/2$ and $|x_1'(t) - x_2'(t)| < c$ for some small value of c. In our numerical computations we ha taken $c = 10^{-6}$. The second special characteristic is that at the moment the sticking region is attained the trajectory of the system is tangent to the switching surface. Thus, for the surface $g_1(y) = y_1 - y_2 - d/2 = 0$, the gradient vector is $\nabla g_1 = (1, -1, 0, 0)^T$ and the sticking region is attained at a point $y^* = (y_2 + d/2, y_2, y_4, y_4)^T$. At this point, the vector field has a value $f(t, y^*) = (y_4, y_4, -(k_1/m_1)y_1, -(k_2/m_2)/y_2)^T$ and clearly, $\nabla g_1(y^*) \cdot f(t, y^*) = 0$. This means that the numerical approximation can not be reliable, in the sense that a small error in the approximation of the solution can lead to a great error in the time t at which the solution starts the sticking region.

A Matlab code to integrate this problem with `disode45` can be

```
%
%  Call to disode45
%

 options=disodeset('RelTol',1.e-5,'AbsTol',1.e-5, ...
                        'ActionSwitch',@actionatswitch);
 y0 = [0.2;0.3;0;0];
 [tout,yout,tdis,ydis,idis,stats]=disode45(@fun, @gfun, ...
                                            [0,10], y0,options);
%
%  Definition of the vector field
%
 function f=fun(t,y)
    m1=2;
    m2=1;
    r1=0.6;
    r2=0.6;
    k1=30;
    k2=20;
    a0=30;
    b0=35;
    d=1.0;
    ee=0.7;
    w=1.38;
    u1=-(r1/m1)*y(3)-(k1/m1)*y(1)+b0/m1+(a0/m1)*cos(w*t);
    u2=-(r2/m2)*y(4)-(k2/m2)*y(2);
    if y(1)-y(2)>= d/2 && u1>=u2,
       f=[y(3);y(3); ...
          -((r1+r2)/(m1+m2))*y(3)-((k1+k2)/(m1+m2))*y(1)+ ...
```

```
          b0/(m1+m2)+(k2*d/(2*m1+2*m2))+(a0/(m1+m2))*cos(w*t);...
          -((r1+r2)/(m1+m2))*y(3)-((k1+k2)/(m1+m2))*y(1)+ ...
          b0/(m1+m2)+(k2*d/(2*m1+2*m2))+(a0/(m1+m2))*cos(w*t)];
    elseif y(2)-y(1)>= d/2 && u2>=u1,
       f=[y(3);y(3); ...
          -((r1+r2)/(m1+m2))*y(3)-((k1+k2)/(m1+m2))*y(1)+ ...
          b0/(m1+m2)-(k2*d/(2*m1+2*m2))+(a0/(m1+m2))*cos(w*t);...
          -((r1+r2)/(m1+m2))*y(3)-((k1+k2)/(m1+m2))*y(1)+ ...
          b0/(m1+m2)-(k2*d/(2*m1+2*m2))+(a0/(m1+m2))*cos(w*t)];
    else
       f=[y(3);y(4);-(r1/m1)*y(3)-(k1/m1)*y(1)+ ...
          b0/m1+(a0/m1)*cos(w*t);-(r2/m2)*y(4)-(k2/m2)*y(2)];
    end
 end
%
%  Definition of the switching surfaces
%
 function [g,isterminal,direction]=gfun(t,y)
    m1=2;
    m2=1;
    r1=0.6;
    r2=0.6;
    k1=30;
    k2=20;
    a0=30;
    b0=35;
    d=1.0;
    ee=0.7;
    w=1.38;
    u1=-(r1/m1)*y(3)-(k1/m1)*y(1)+b0/m1+(a0/m1)*cos(w*t);
    u2=-(r2/m2)*y(4)-(k2/m2)*y(2);
    g=[y(1)-y(2)-d/2;y(2)-y(1)-d/2;u1-u2];
    isterminal=[-1;-1;0];
    direction=[1;1;0];
 end
%
%  Definition of the action at switch function
%
 function ysw=actionatswitch(t,y)
    m1=2;
    m2=1;
    r1=0.6;
    r2=0.6;
    k1=30;
    k2=20;
    a0=30;
```

```
b0=35;
d=1.0;
ee=0.7;
w=1.38;
ysw=y;
if abs(y(3)-y(4))>1.e-6,
   ysw(3)=((m1-m2*ee)/(m1+m2))*y(3)+((1+ee)*m2/(m1+m2))*y(4);
   ysw(4)=((m2-m1*ee)/(m1+m2))*y(4)+((1+ee)*m1/(m1+m2))*y(3);
   if y(1)-y(2)-d/2>=0
      ysw(1)=y(2)+d/2;
   elseif y(2)-y(1)>=d/2,
      ysw(2)=y(1)+d/2;
   end
else
   ysw(3)=ysw(4);
   if y(1)-y(2)>=d/2
      ysw(1)=y(2)+d/2;
   elseif y(2)-y(1)>=d/2,
      ysw(2)=y(1)+d/2;
   end
end
```

Once the integration of this problem concludes, the vectors `tdis`, `ydis` and `idis` contain the data corresponding to 135 transversal discontinuity points, 3 points starting a sticking region and two exits of a sticking region. The next matlab code give the data corresponding to the sticking points, for which `idis` is negative.

```
>>tdis(idis<0)

ans =

    1.1658    1.4446    3.6106    6.1162    8.3541

>> ydis(idis<0)

ans =

    0.9245   -0.1332    1.3458    0.2212    1.4245
```

The plots in Figure 5 (without legends) can be obtained with the matlab orders

```
plot(tout,yout(:,1),'k',tout,yout(:,2),'b',tdis(idis<0), ...
```

```
        ydis(idis<0,1),'ro',tdis(idis<0),ydis(idis<0,2),'ro')
plot(tout,yout(:,1)-yout(:,2))
```

Example 4 In this example, we consider a simplified model of structural pounding, used in the study of the effects of earthquakes [3]. It is defined by the second order equation

$$2x'' = -4.1\, x' - 210.125\, x - u(x, x') - r(t), \qquad t \in [0, 3]$$

with $r(t) = 2\sin(14\, t)$ and u given by

$$u(y, y') = \begin{cases} 0 & \text{if } x < \nu, \\ c \cdot (x-\nu)^{\frac{3}{2}} \ + 1.98\sqrt{2c}(x-\nu)^{\frac{1}{4}}\, x' & \text{if } x > \nu,\ x' > 0, \\ c \cdot (x-\nu)^{\frac{3}{2}} & \text{if } x > \nu,\ x' < 0, \\ c = 2.47 \times 10^6, \quad \nu = 0.005. & \end{cases}$$

Expressed as a first order system with two components $y_1(t) = x(t)$, $y_2(t) = x'(t)$, we have

$$y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ -4.1\, y_2 - 210.125\, y_1 - u(y_1, y_2') - r(t) \end{pmatrix} = f(t, y)$$

Clearly the vector field $f(t, y)$ is non smooth at points where $x(t) = y_1(t) = \nu$ or where $x(t) = y_1(t) > \nu$ and $x'(t) = y_2(t) = 0$. Then, we have two switching surfaces $g_1(y) = y_1 - \nu$ and $g_2(y) = y_2$. For the switching surface g_2, the vector field is discontinuous only when x' changes from positive to negative, which happens when $x > \nu$. Moreover, due to the powers 1/4 and 3/2, the function defining the vector field at the region $g_1(y) > 0$ is not defined when $g_1(y) < 0$.

Note that the vector field f is a continuous function (but not $\mathcal{C}^1$). Therefore $f_+(t_d, y_d) = f_-(t_d, y_d)$ at the switching points and the transversality condition is satisfied unless the vector field is tangent to the switching surface. It is easy to see that $\nabla g_1(y) \cdot f(t, y) = 1$ for all y and $\nabla g_2 \cdot f(t, y) = -210.125 y_1 - c \cdot (y_1 - \nu)^{3/2} - r(t)$ for switching points such that $y_2 = 0$. Consequently, the transversality condition is satisfied except for the points for which $y_2 = x' = 0$, $y_1 = x > \nu$ and

$$-210.125\, x(t) - c\, (x(t) - \nu)^{3/2} - r(t) = 0.$$

Since $|r(t)| \leq 2$, whenever $x(t) \notin [\nu, 0.0051157248]$ the discontinuity points are transversal.

The phase diagram for this problem (x_1 versus x') is depicted in Figure 6

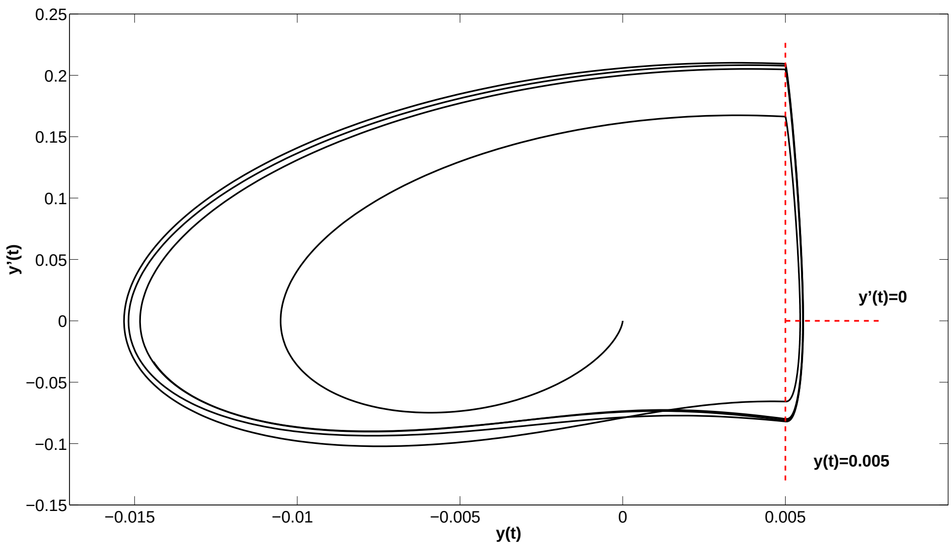


Figure 6: Phase diagram for structural pounding model

The discontinuity points (where the third derivative $x'''(t)$ is not continuous) lie onto the red dashed lines.

A Matlab code to integrate this problem with `disode45` can be

```
%
%  Call to disode45
%
 options=disodeset('Refine',10);
 y0 = [0;0];
 [tout,yout,tdis,ydis,idis,stats]=disode45(@fun4, ...
                                   @gfun4,[0,3], y0,options);
%
%  Definition of the vector field
%
 function f=fun(t,y)
      k=210.125;
      c=2.47e+6;
      nu=0.005;
      r=2*sin(14*t);
      if y(1)>nu
        if y(2)>0
           u=c*(y(1)-nu)^(3/2) + 1.98*sqrt(2*c*sqrt(y(1)-nu))*y(2);
```

```
        else
           u=c*(y(1)-nu)^(3/2);
        end
      else
        u=0;
      end
      f=[y(2); (-4.1*y(2)-k*y(1)-u-r)/2];
 end
%
%  Definition of the switching surfaces
%
 function [g,isterminal,direction]=gfun(t,y)
    if y(1)<0.005
       g=[y(1)-0.005;1];
    else
       g=[y(1)-0.005;y(2)];
    end
    isterminal=[0;0];
    direction=[0;-1];
 end
```

Once the integration of this problem concludes, the vectors tdis, ydis and idis contain the following data

```
tdis =

    0.4006    0.4071    0.4172    0.8384    0.8446    0.8543
    1.2818    1.2880    1.2977    1.7307    1.7368    1.7467
    2.1798    2.1860    2.1958    2.6286    2.6348    2.6446


  Columns 13 through 18

    2.1798    2.1860    2.1958    2.6286    2.6348    2.6446

ydis =

    0.0050    0.1664
    0.0055    0.0000
    0.0050   -0.0658
    0.0050    0.2095
    0.0055    0.0000
    0.0050   -0.0820
    0.0050    0.2079
    0.0055   -0.0000
    0.0050   -0.0811
```

```
    0.0050    0.2048
    0.0055    0.0000
    0.0050   -0.0799
    0.0050    0.2047
    0.0055    0.0000
    0.0050   -0.0799
    0.0050    0.2049
    0.0055    0.0000
    0.0050   -0.0799


idis =

     1     2     1     1     2     1     1     2     1
     1     2     1     1     2     1     1     2     1
```

The plot in Figure 6 (without legends) can be obtained with the matlab order

```
plot(yout(:,1),yout(:,2),[0.005,0.005],[-0.15,0.25],'r--', ...
                                      [0.005, 0.008],[0,0],'r--');
```

Example 5 A bouncing ball model (ODE with impulses) is governed by the equation

$$x'' = -9.8.$$

Starting from a height $x(0) = x_0$ with velocity $x'(0) = x'_0$, the ball falls and when it attains the floor $x(t_d) = 0$, with velocity $x'(t_d) = x'_d$, the integration must be restarted with initial conditions $x(t_d) = 0$, $x'(t_d) = -0.9\, x'_d$ where the factor 0.9 represents the lost of energy.

Expressed as a first order system with two components $y_1(t) = x(t)$, $y_2(t) = x'(t)$, we have

$$y' = \begin{pmatrix} y'_1 \\ y'_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ -9.8 \end{pmatrix} = f(t, y)$$

Here the switching surface is clearly $g(y) = y_1$.

In this case the discontinuity affects the solution, and the jump of the state must be provided in some way by the user. In the code this is done by means of an external function `actionatswitch(t,y)` that from the current state y_d provides the new state y_d^*.

A Matlab code to integrate this problem with `disode45` can be

```
%
 options=disodeset('AbsTol',1.e-4,'RelTol',1.e-4, ...
                'Refine',10, 'ActionSwitch',@actionatswitch);
 y0=[10; 0];
 [tout,yout,tdis,ydis,idis,stats]=disode45(@fun, ...
                             @gfun,[0,20], y0, options);
%
%  Definition of the vector field
%
 function ydot=fun(t,y)
      ydot=[y(2); -9.8];
 end
%
 function [g,isterminal,direction]=gfun(t,y)
      g=y(1);
      isterminal=-1; %  Call to gwhenswitch when a switching point is found
      direction=-1;
 end
 %
 %  Output switch function
 %
 function ysw=actionatswitch(t,y)
       ysw=[0; -0.9*y(2)];
 end
```

Once the integration of this problem concludes, the vectors `tdis`, `ydis` and `idis` contain the data corresponding to 13 switching points.

```
tdis =

     1.4286    4.0000    6.3143    8.3971   10.2717   11.9588   13.4772
    14.8438   16.0737   17.1806   18.1768   19.0734   19.8804

ydis =

    0.0000  -14.0000
   -0.0000  -12.6000
   -0.0000  -11.3400
   -0.0000  -10.2060
   -0.0000   -9.1854
    0.0000   -8.2669
    0.0000   -7.4402
    0.0000   -6.6962
   -0.0000   -6.0265
    0.0000   -5.4239
```

```
-0.0000   -4.8815
-0.0000   -4.3933
-0.0000   -3.9540
```

Example 6 A heating model (problems with a switching vector field). The variation of the temperature in a room is supposed to be linear on the difference $x(t) - T_{\text{ext}}$ between the current temperature $x(t)$ and the external temperature T_{ext}. Initially, if the temperature $x(0)$ is lower than a maximum temperature $T_{\max}$, the heating is on, a constant external heat source u is acting and the temperature satisfies the equation

$$x' = -K \cdot (y - T_{\text{ext}}) + u,$$

until the temperature attains $T_{\max}$. At that moment the heat source u is set off and the differential equation changes to

$$x' = -K \cdot (y - T_{\text{ext}}).$$

The heat source is set on again when $x(t) = T_{\min}$.

Note that in this example the vector field itself is modified from $f(t,x)$ to a different one $\hat{f}(t,x)$ when the solution attains certain point t_d satisfying a condition $g(t_d, x(t_d)) = 0$. For the same point (t,x), the vector field $f(t,x)$ can not be equal to $\hat{f}(t,x)$.

In order to solve this problem with DISODE45, it must be transformed, by adding an additional equation, into an equivalent problem that can be treated as an ODE with impulses. Defining $y(t) = (y_1(t), y_2(t))^T \equiv (x(t), y_2(t))^T$, where $y_2(t)$ is going to be a constant, the heating can be modelled by

$$\begin{aligned}
&y_1' = \begin{cases} -K \cdot (y_1 - T_{\text{ext}}) + u & \text{if } y_2(t) = 1, \\ -K \cdot (y_1 - T_{\text{ext}}) & \text{if } y_2(t) = -1, \end{cases} \\
&y_2' = 0, \\
&y_1(0) = x(0) = x_0, \quad y_2(0) = \begin{cases} 1 & \text{if } x_0 < T_{\max}, \\ -1 & \text{if } x_0 \geq T_{\max}. \end{cases}
\end{aligned}$$

There are two switching surfaces $g_1(y) = y_1 - T_{\max}$ and $g_1(y) = y_1 - T_{\min}$. When the solution attains the first one from temperature $x(t) < T_{\max}$, or the second one from temperature $x(t) > T_{\min}$, the sign of the variable $y_2(t_d)$ is changed.

A Matlab code to integrate this problem with `disode45` can be

```
%
 options=disodeset('AbsTol',1.e-4,'RelTol',1.e-4,'ActionSwitch',@actionatswitch);
 y0=[15;1];
```

```
 [tout,yout,tdis,ydis,idis,stats]=disode45(@fun, @gfun,[0,20], y0, options);
%
 function ydot=fun(t,y)
     if y(2)==-1
       ydot=[-0.1*(y(1)-18); 0] ;
     else
       ydot=[-0.1*(y(1)-18)+2; 0] ;
     end
 end
%
 function [g,isterminal,direction]=gfun(t,y)
      g=[y(1)-23.5; y(1)-22];
      isterminal=[-1;-1];  %  Call to actionatswitch when found
      direction=[1;-1]; % From negative to positive the first one
 end
 %
 %  Output switch function
 %
 function yswitch=actionatswitch(t,y)
     if y(2)==1,
       yswitch=[y(1); -1];
     else
       yswitch=[y(1); 1];
     end
 end
```

Once the integration of this problem concludes, the vectors `tdis`, `ydis` and `idis` contain the data corresponding to 13 switching points.

```
tdis =

    4.6135    7.7980    8.7824   11.9669   12.9513   16.1359   17.1203

ydis =

   23.5000    1.0000
   22.0000   -1.0000
   23.5000    1.0000
   22.0000   -1.0000
   23.5000    1.0000
   22.0000   -1.0000
   23.5000    1.0000

idis =
```

1 2 1 2 1 2 1

References

[1] V. Acary and B. Brogliato, *Numerical Methods for Nonsmooth Dynamical Systems* (Lecture Notes in Applied and Computational Mechanics, Springer–Verlag, Berlin, 2008).

[2] M. Calvo, J.I. Montijano and L. Rández, *DISODE45, A Matlab Runge-Kutta solver for Piecewise Smooth IVPs of Filippov type*, to appear in ACM TOMS.

[3] R. Jankovski, *Non-linear viscoelastic modelling of earthquake-induced structural pounding*, Earthq. Eng. Struct. Dyn. 34(2005), pp. 595-611.

[4] A.C.J. Luo, *Discontinuous Dynamical Systems on Time-varying Domains* (Nonlinear Physical Science, Springer–Verlag, Berlin, 2009).