1. ORGANIZATION AND USAGE

A makefile is provided to compile the codes in this package. After unpacking this software, simply execute the command

```
build
```

in the main directory MixedVol to generate the executable code "mixedvol".

1.1 Call Sequence and Output

The software package MixedVol is easy to use. To calculate, for example, the mixed volume of the support $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$, where

$$\begin{aligned} \mathcal{S}_1 &= \{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}, \\ \mathcal{S}_2 &= \{(0,0,0),(2,2,2)\}, \\ \mathcal{S}_3 &= \{(0,0,0),(3,0,0),(0,3,0),(0,0,3)\}, \end{aligned}$$

a file should be created to contain the support data in the following format:

```
3
5
2
4
0  0  0
1  0  0
0  1  0
0  0  1
1  1  1
0  0  0
2  2  2
0  0  0
3  0  0
0  3  0
0  0  3
```

The number on the first line is the dimension of the support. For this support it is 3. The numbers from line two to line four give the numbers of points in the individual support $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$. They are 5, 2 and 4 points, respectively. The next five lines contain the points in $\mathcal{S}_1$ with each point listed on one line. The two lines after those contain the points in $\mathcal{S}_2$. The last four lines are the points in $\mathcal{S}_3$. Call this file MySupport, say, and execute the command

```
mixedvol -s MySupport
```

It will produce the mixed volume of the support (namely 18) and will output it on screen. The flag "`-s`" means the input file consists of a support. Note that for an n-dimensional support $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$, all information about $\mathcal{S}_1, \ldots, \mathcal{S}_n$ must be listed in the file. The algorithm will automatically determine the semi-mixed structure of the support when it exists.

To calculate, for example, the mixed volume of the following system of 3 polynomials in 3 variables,

$$\begin{aligned} p_1(x_1,x_2,x_3) &= (x_1^2+x_2^2+x_3^2-1)(x_1-0.5)(x_2-x_1^2), \\ p_2(x_1,x_2,x_3) &= (x_1^2+x_2^2+x_3^2-1)(x_2-0.5)(x_3-x_1^3), \\ p_3(x_1,x_2,x_3) &= (x_1^2+x_2^2+x_3^2-1)(x_3-0.5)(x_3-x_1^2)(x_2-x_1^2), \end{aligned}$$

a file should be created first to define the polynomials of the system in the following format:

```
{
 (x1^2+x2^2+x3^2-1)*(x1-0.5)*(x2-x1^2);
 (x1^2+x2^2+x3^2-1)*(x2-0.5)*(x3-x1^3);
 (x1^2+x2^2+x3^2-1)*(x3-0.5)*(x3-x1^2)*(x2-x1^2);
}
```

The file starts with a "{" and ends with a "}". Each polynomial ends with a ";". Call this file MyPolynomials, say, and execute the command

```
mixedvol -p MyPolynomials
```

It will produce the mixed volume of the polynomial system (namely 119) and will output it on screen. The flag "`-p`" means the input file consists of a polynomial system.

1.2 General Description of the Modules

To calculate the mixed volume of a support $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$, $\mathcal{S}_i \subset \mathbb{Z}^n$ for $i = 1, \ldots, n$, the main module MixedVolDriver calls two modules Pre4MV and MixedVol.

The module Pre4MV determines all non-vertex points of each support $\mathcal{S}_1, \ldots, \mathcal{S}_n$. It permanently removes them from the mixed volume computation because they don't contribute to the mixed volume of the support, resulting in a smaller support. Then Pre4MV determines the semi-mixed structure of the smaller support and rearranges the support.

From the output of Pre4MV, if the support $\mathcal{S}$ has the semi-mixed structure $(\mathcal{S}^{(1)}, \ldots, \mathcal{S}^{(r)})$ of type $(k_1, \ldots, k_r)$, i.e., each $\mathcal{S}^{(i)}$ is repeated k_i times for $i = 1, \ldots, r$, the module MixedVol then computes all mixed cells of type $(k_1, \ldots, k_r)$ by searching through the lower hull of the randomly lifted support $(\hat{\mathcal{S}}^{(1)}, \ldots, \hat{\mathcal{S}}^{(r)})$, where

$$\hat{\mathcal{S}}^{(i)} = \{(a, \omega_i(a)) \mid a \in \mathcal{S}^{(i)} \text{ and } \omega_i(a) \text{ is a random real number}\}, \quad i = 1, \ldots, r.$$

A mixed cell $(C_1, \ldots, C_r)$ of type $(k_1, \ldots, k_r)$, where for each $i = 1, \ldots, r$, C_i consists of $k_i + 1$ affinely independent points of $\mathcal{S}^{(i)}$, is defined to have the property

$$\dim(\text{conv}(C_1 + \cdots + C_r)) = \dim(\text{conv}(C_1)) + \cdots + \dim(\text{conv}(C_r)) = k_1 + \cdots + k_r = n.$$

And the mixed volume of the support $(\mathcal{S}^{(1)}, \ldots, \mathcal{S}^{(r)})$ of type $(k_1, \ldots, k_r)$ can be assembled from the volumes of all the mixed cells of type $(k_1, \ldots, k_r)$ induced by $(\hat{\mathcal{S}}^{(1)}, \ldots, \hat{\mathcal{S}}^{(r)})$.

To calculate the mixed volume of the support of a system of n polynomials in n variables, the main module MixedVolDriver first calls the module PolynomialSystemReader, which reads the polynomial system from an input file and generates its support. With the support of the polynomial system available, the main module MixedVolDriver then calls the modules Pre4MV and MixedVol to calculate the mixed volume of the polynomial system. The module PolynomialSystemReader is developed in [Li and Li (2001)]. The mixed cells together with the lifting function $\omega = (\omega_1, \ldots, \omega_r)$ of the support are crucial for solving the polynomial system by the polyhedral homotopy continuation method.

In this release of the package MixedVol, the module PolynomialSystemReader together with its dependencies are grouped into the sub-directory "SRC/PolyReader", the module Pre4MV as well as its dependencies are located in the sub-directory "SRC/PreProcess", and the module MixedVol along with its dependencies are in the sub-directory "SRC/MixedVol".

REFERENCES

Li, T.Y. and Li, X. 2001. Finding mixed cells in the mixed volume computation. *Foundation of Comput. Math.* **1**, 161–181. Software package available at http://www.math.msu.edu/~li.