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Committee on Mathematics Education

Reference Levels in School Mathematics Education in Europe

Niveaux de référence pour l'enseignement des mathématiques en Europe

International Report

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1. Introduction

The study "Reference levels in School Mathematics Education in Europe at the age of 16, as it was suggested to the European Commission (EC, D.G.XXII) plans to identify « Reference Levels » concerning knowledge and competencies in the domain of mathematics that can become common to all countries in the European Union, and perhaps in other countries.

The study is conducted under the auspices of the Committee on Mathematics Education of the European Mathematical Society (EMS). It is conducted, in part, by mathematicians, or teachers of mathematics at university level who are particularly concerned about the future of mathematical education.

A Group of Experts was constituted as an extension of the Committee on Mathematical Education of the EMS. This ensures the Group's output both recognition from the national mathematical societies and the means of circulation for this output.

The Group formed around members of the Committee includes teachers of secondary education, researchers on mathematical education and on the teaching and learning processes (didactics of mathematics), teacher trainers and authors of curricula or textbooks for secondary teaching.

Mathematicians from the Group, in all cases, are highly interested in the teaching and learning of mathematics at school, they are well informed about the state of the art in their respective countries, and have privileged relations with their national associations of mathematicians and mathematics teachers.

The coordinating centre of the study was set up in France, at the IREM (Institut de Recherche sur l'Enseignement des Mathématiques de l'Université de Franche-Comté) in Besançon under the responsibility of Antoine BODIN, researcher at the IREM and the INRP (Institut National de Recherches Pédagogiques). The IREM in Besançon has a long-standing tradition of research in the evaluation of mathematics and in participating in national (the French Ministry of Education) and international (TIMSS) studies. For more than 10 years in France, the Besançon IREM has been involved in developing a continuous evaluation unit of the teaching of mathematics, which is known as EVAPM. This unit is organised by the "Association des Professeurs de Mathématiques de l'Enseignement Public (APMEP)", the French Organisation of Mathematics Teachers (from primary to university level)" in partnership with the INRP (Institut National de Recherches Pédagogiques), the National Institute for Research in Education, and the Besançon IREM. The evaluations on a large scale within the framework of EVAPM secured, at least for France, the availability of referential data on the knowledge and competencies of the students.

After a first Committee meeting hold in Trento (Italy), three meetings were held in Besançon with Professor Vinicio Villani of the University of Pisa (Italy) and President of the Committee on Mathematics Education of the European Society of Mathematics as Chairman ; he is also in charge of the study. The first meeting was held in March 1999, the second one in December of the same year, and the third in February 2001.

Bringing together between 20 and 25 experts and lasting three days, each meeting was an opportunity for intensive and fruitful exchange of information and work. In each of the first two meetings, the participation of the President of the European Mathematical Society in these meetings demonstrates the interest of the mathematical community in the work of the Group. The second section of the present report contains a synthesis of the considerations and the debates during these meetings.

Naturally, numerous considerations on these matters already exist following inquiries or national and international studies such as those from OECD and IEA (TIMSS, ...). However, none of these studies cover all of the countries within the European Union, and only a few of them, in one form or another, examine the population of students of 16 years of age. In addition to this, educational systems develop rapidly and numerous collected data from the last 10 years (and even over the last 3 or 4 years) is either totally obsolete or has to be verified before being used. On the other hand changes are taking place everywhere and it would be dangerous to claim to offer a stabilised portrait.

Finally this study will naturally have set up a European network of persons and organisations interested in the question, whose main agents are the experts of the Working Group itself.

2. Evolution of the study : from objectives to results

The development of exchanges of all types between the different countries of the European Union implies that common bases for dialogue and reflexions be developed in a large number of areas. The teaching of mathematics is one of these fields. The European Mathematical Society - with its Committee for Mathematical Education - is a partner that is naturally concerned by these exchanges. These exchanges, whose need is being felt more and more, may concern both questions of general interest (the place of mathematical education in the life of the citizen, the impact of mathematical modelisation in the taking of decisions…) and specific questions (methodology of mathematical teaching, programmes of research or teacher training, special disciplines, the place of Information Technology…). The study ‘Reference Levels in School Mathematics Education in Europe at the age of 16’ falls in this framework.

Without dwelling too long on the existing situation or on other studies currently in progress, the Working Group could have tried to define what would seem desirable or necessary in terms of knowledge and competencies for all young people of 16 years old in the European Union. It would in doing this have built a fine intellectual edifice which would no doubt have been extremely satisfying for the intellect but would have run the risk of proving virtual or, perhaps even worse, normative.

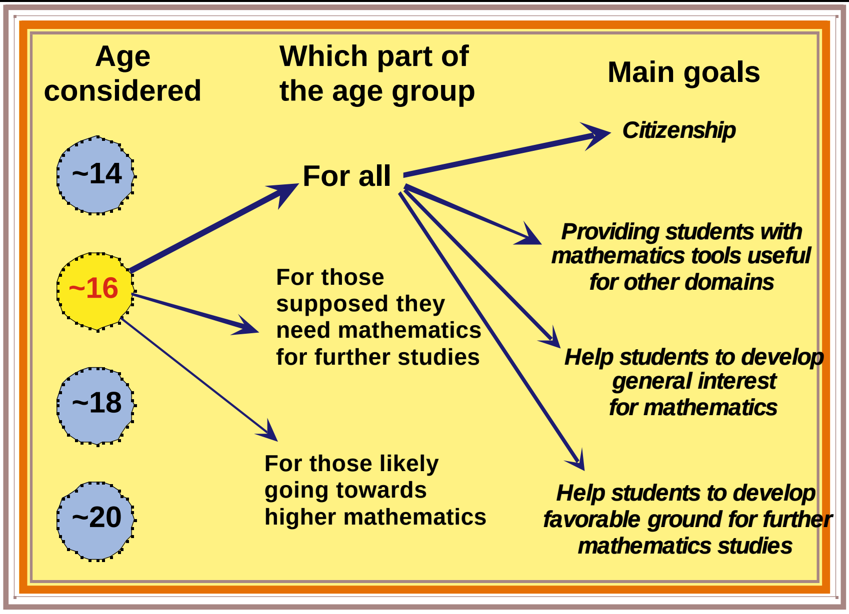
Aware of this difficulty, the members of the Group decided to begin the study by examining the structural aspects of the educational systems of the countries participating in the study, paying special attention to those aspects which are important and influential on mathematics teaching and learning.

It was clearly affirmed that the study was about the contents and the methods of teaching (forms of learning), and that it does not aim at standardising the curricula, but that it should establish or develop links between secondary and university (post-secondary) teaching.

In this respect, a first, minimal classification of knowledge and expected competencies was accepted, namely :

Level 1 : for the correctly informed citizen (i.e. everybody),

Levels 2 and 3 : for those who continue on to post-secondary studies.



Position de l'étude

L'épaisseur des flèches correspond à l'importance considérée à 16 ans

Cette importance relative ne serait pas la même à 14, 18 ou 20 ans

2.1. The Notion of Reference Level

Initially the Working Group decided to examine in depth official documents (teaching curricula, school books…) corresponding to the age-group of approximately 16 years. And in a more pragmatic vein, it wanted to start by observing the current state of affairs by looking into the following questions :

- How do the educational systems of the countries participating in our study function today, and more specifically, the parts related to mathematics?
- Concerning mathematics education : what are the strong and weak points of these systems as pointed out by the persons involved, or by national / international studies or inquiries?
- At present, what are the explicitly stated expectations in the field of mathematics at the age of 16 in the countries of our study ? And then, with regard to these expectations, what is the actual level of knowledge and what are the actual competencies observed among the students?

The replies to these questions allowed a better understanding of what presently :

- is supposed to be taught (within the official curriculum)
- is in fact taught and how it is taught (contexts, teaching methods - the "real" curriculum)
- is assessed and how it is assessed (continuous assessment or formative evaluation, examinations)
- is learned by the students (attained curriculum)
  - in mathematics
  - in secondary education
  - at the end of compulsory schooling and in every member state of the European Union.

The fundamental objective of the project, that is to say **defining and fine tuning a common base of information and considerations concerning the teaching of mathematics** was therefore able to begin to take shape.

As a result of this first phase of reflection, the various members of the Group undertook to produce a **National Report** concerning the teaching of mathematics in their country.

In addition this first phase also enabled a relatively free definition of the **Reference Level** as a pragmatic **basis for information sharing, communication and dialogue.** In a more detailed way, the definition of Reference Levels was considered to be the description – whether in general or more circumstantial terms – and for each country, of the subjects or of the curriculum, of the degree of depth of treatment (their impact on the mathematical culture of the students) and of the students' mode of integrating this (this latter having a strong connection with the targeted competencies). It was therefore decided to attempt to define the targeted subjects or competencies and – proceeding with a guideline of reality –which had been more or less well assimilated according to the individuals.

In such a context, it is important to ensure that no meaning that is vaguer or subject to differing interpretations be linked to 'Reference Level' in which the two terms of 'Level' and 'Reference' might receive too negative an interpretation (especially with teachers) ; particularly so as these terms embody complex technical and didactic problems and in addition also have a political connotation. Should one class individuals according to criteria defined by a higher administrative body, a supranational one at that? Should one control at this level the decisions taken at the local level : for instance the attribution of local diplomas? Should one furthermore control the quality of the education meted out in the Member States of the European Union?

2.2. National Reports (a brief overview)

It was agreed that each national report should focus not only on the teaching of mathematics for students of 16 years old but also on its context with the idea of trying to include 100% of the young adults of this age-group wherever they are studying and whatever their future studies might be.

Even if the principal aim of the study is to give references that can be useful with a focus on the age-group of 16 year-olds, we were convinced of the impossibility of limiting the study to this age.

With respect to age, 16 years is more of a reference than the actual age taken into consideration.

It is clear that :

- Depending on the country, students 16 years of age can all (or nearly all) be in the same grade, especially in countries where the repeating of a year is unknown (like in Sweden). In other countries, students 16 years of age can be in different grades.
- Again, depending on the country, students of 16 years of age may all be practically in one type of school, or, at least, may have been in one and only one type of school ; whereas in other countries, they may have already been in different types of schools for several years.

Whatever the situation is, and in order to be relevant for a given age-group, it is necessary to have information on what has happened during the course of the previous years and what will happen in the years to come. The simple fact that a certain concept or element of knowledge is not taught before the age of 16 is different in terms of relevance if it is certain that the student has a small chance of encountering it later on, or on the contrary, if it is certain that he/she is to be confronted with it later.

In principle, we were also interested in knowing where the students of 16 years of age had actually reached in each country and what they were learning mathematically at this age regardless whether they are or are not in the class theoretically appropriate for this age (if this concept exists).

Concerning the countries ,the study did not only include members of the European Union. At present (May 11, 2001), National Reports are available form from the following countries : **Belgium, Denmark, Finland, France, Germany, Greece, Hungary, Italy, Luxembourg, Netherlands, Poland, Russian Federation, Spain, Sweden, Switzerland, United Kingdom.**

Concerning countries, the study has been extended to some East European countries (Russia, Poland, Hungary) as well as Switzerland (reciprocal interest). Only three countries of the European Community are absent of the study (Austria, Eire, Portugal).

The following structure was then agreed upon for the national presentations :

1. General description of the context of mathematics teaching and learning
2. Principal aims accepted for mathematics teaching and learning
3. Basic contents : a comprehensive and commented inventory
4. Examples of topics to be taught and learned (quadratic equations, theorem of Pythagoras, similitude, percentages,..)
5. Additional aspects of teaching and learning mathematics (regional characteristics, ….strategies, initial training and further training of teachers, didactic resources…).

Section number 5 of the present report summarises the National Reports. The reading of these reports that are annexed will allow those who so wish to find all the details of interest to them.

2.3. «Dream Questions» (reference questions)

The idea of « Dream Questions » *(reference questions)* is obviously associated with knowledge evaluation (cf. Bodin, 2000). Yet the Working Group estimated that, before dealing with evaluation itself, it would be important to examine the quality of teacher training and to facilitate communication and diffusion, within Europe and in other countries, of mathematical teaching practices liable to improve this quality.

So the Group applied itself to identify, to produce and to analyse mathematical questions (exercises and problems) illustrating what was considered as pertinent for the teaching of mathematics for students of 16 years old (and around this age).

In other words, the idea was to collect a bunch of situations we would like the students to be confronted with during their studies, without wishing them to be integrated into any set of questions for purposes of evaluation.

Some of the questions selected are even representative of classroom subjects to be used preferably with whole classes or in small groups rather than for individual evaluation situations.

In order to characterise the voluntaristic nature of this approach which entails in fact, and after careful analysis on our part, presenting what seemed suitable to us, even if this is sometimes far removed from usual practices observed in a number of countries or in strata of educational systems, we called our questions « dream questions », translated in the French version of this report by «questions de rêve ».

In spite of this, we have tried to remain realistic. All the questions presented have actually been used with students. The trial has however been limited and a validation exercise is still to be carried out in this field.

One of the aims of the project was to collect and to complete a set of questions and problems in order to :

1. Be able to offer a selected list of questions and problems chosen for their cultural and educational qualities (involving the capacity of mathematisation, of reasoning, of interpreting results…). These qualities are explicited and analysed.
2. Distinguish among these questions and problems, those which focus more on training strategies (classroom situations) and those which are more likely to be used for evaluation (i.e. to detect knowledge acquired by the students).
3. Offer the teaching material produced to all who are ready to use them, with however no intention of reducing curricula to an unique model.

With this object in mind, besides the questions produced by our Working Group, we were able to borrow elements from several national or international studies.

For reasons of communication, the theoretical organisational framework of the questions relative to the concerned content and the competencies required was partially borrowed from the PISA study carried out by the OECD. We chose a selection of 65 questions from this framework.

The reference questions and theoretical framework are presented in the second part of this report.

It would of course be desirable to continue and examine how to :

1. Give, for each problem, a scale to show all possible student answers, the value of these answers and, in the case of evaluation questions, percentages of success already scored in the various countries.
2. Produce documents on efficient didactical routes (organisation of training situations) in order to improve the general teaching given to students (and not only for training them in solving the problems in the tests).

This could be the aim of a second phase.

In order to achieve the aims as defined above, it was necessary to collect some information on the teaching and learning of mathematics in the countries concerned by the study.

At the Besançon IREM and in relation to the study, we built up a documentation centre on mathematics curricula and examinations. Even if it is small, this centre is now able to receive and organise the documentation needed to initiate necessary translations etc. In addition to this, it is equipped with the means to facilitate the exchange of information, initially among the experts of the study, and later on a larger scale, to include the national mathematical and teacher associations.

A diversity of information is gathered at the centre :

- General information on the educational systems and more particularly on the teaching and learning of mathematics : official curricula (including information on examinations and evaluations, textbooks, copies of exams and tests for evaluation and a diversity of reports).
- Special information on the topics taught and the methods of teaching.
- Documents for teachers.

Each national expert was requested to provide information of this type concerning his/her country to the Resource Centre. To do that, he/she had to contact official institutions, teacher organisations, editors of textbooks and, if available, editors of evaluation tests, as well as institutions having done research on mathematics education (didactics), pertinent projects or evaluations and all other projects on the teaching and learning of mathematics.

Of course, special interest was always given to information about teaching and learning in the 16 age bracket , but this was not exclusive.

Simultaneously, the centre gathered information collected from other inquiries and international studies like TIMSS and PISA… It also seemed desirable to include developments happening in other countries such as the USA and Japan. In particular, it seemed essential to make a comparison between our propositions and the "Standards" developed in the USA for instance. Furthermore, it should be remembered that there is an international mathematics community that is well organised and constantly in touch with questions concerning the teaching of mathematics. The International Congress on Mathematical Education (ICME) which meets every 4 years is one highly visible manifestation of this. The ideas, the reflections and the exchanges on the teaching of mathematics go well beyond national borders and it would be both a pity and difficult to lock oneself into a purely European consideration. In this way, the documentation gathered at Besançon is not limited to the documentation of the countries included in the study but has widened to other sources.

3. General Considerations on Society and Educational Systems

3.1. State of Educational Systems

In the introduction, we have already hinted at the fact that changes are taking place all over the European Union. Consequently, it is a bit difficult to claim to offer a stable picture of the actual state of the different educational systems.

The centralised systems are tending to be decentralised while decentralised countries are tending to re-establish a more central leadership. In addition to this, a survey of the OECD shows that in both cases, theses developments are in accordance with the wishes of the citizen.

The idea of »accountability» is of growing importance, and systems all over Europe, even if in highly different ways, are trying to develop the means with which to evaluate the effects of their educational systems. This is in concordance with what, in France, is called « the culture of evaluation », an effort to include the question of accountability for as many aspects of the system as possible. This concerns the teaching and learning in the classroom, as well as the organisation of the educational institutions, the quality of curricula and textbooks, and even the learner himself/herself (in the form of self-evaluation).

The changes that are underway try to correspond to the transformation that occurred in the last decades of the twentieth century in all societies of the countries under study. We will further detail three points which are of particular importance for the functioning of the educational systems : the gradual development of education for all, the introduction of new technologies, and growing mobility in a changing labour and education market - in Europe and world-wide.

3.1.1. Education for all

In the future and in all countries under study, there will be nine years of compulsory schooling. In fact, nearly all young adults go to school / college until the age of 16, and in most cases, even until the age of 18.

Nevertheless, our interest in the young adults of 16 years of age leads to a first difficulty. Even within the European Union, the OECD statistics (1996) show, for at least half of the countries, a percentage of 5 to 20% of young adults who do not go to school / college (Austria, Denmark, Finland, Greece, Ireland, Portugal, Spain, Switzerland, United Kingdom).

In contrast to that and again according to the OECD statistics, the (mathematical) expectancy of the duration of schooling at all levels, university levels included, would be to approximately 16 years, and from the age of 4 in more than half of the countries studied. This means that on average, at the age of 16, young adults can no longer be provided with schooling, and hence are out of the reach of our study, whereas all the others may even have an additional four years of training and education.

All this can only be described as tendencies. In reality, things are changing rather quickly at the legal level (the end of compulsory schooling is in the process of being changed from 14/15 years to 16, or even 18 years in some countries), as well as at the factual level. To give an example, two-thirds of young French adults who have finished a period of brief higher education, which provides them with a diploma, continue to study in one way or another. However, these brief and certified studies were set up to give them direct, open access to the working world. This very same tendency can be found in other countries, and this development seems to be linked to the saturation of the labour market (this varies from country to country).

Besides this lengthening of time spent in school, there is a general tendency to postpone as long as possible the differentiation (organisation of subjects and courses) in schools. Generally speaking for the countries under consideration, this differentiation occurs in the majority of cases only before the age of 14 / 15. As for the other « context variables », the dispersion observed for this aspect is significant.

For certain countries, differentiation takes place at the age of 16 and even at 17 (but not of course for the small portion of those who have already left school at this age). Denmark, Finland and Sweden are in this category. Other countries make this differentiation at the age of 11 or 12 : Austria, Belgium, Germany, Netherlands, Switzerland, and the United Kingdom (but some of these countries seem to want to retard this). But we have to stress that all this information may have to be revised in order to take into account developments that may be under way.

3.1.2. Introduction of new information technologies

New information and communication technologies have been introduced in all professional and non-professional domains. They imply new expectations in terms of competencies, training and education.

As a consequence, schools and colleges are being asked to supply basic training in information technology for everyone. Here, we have a new kind of knowledge to challenge « reading, writing, arithmetic » and it is not clear whether it is supposed to supplement or to replace part or all of the former knowledge.

For example ,in some experimental studies concerning writing, which begin very early on in childhood, the notion of « hypertext » tends to be taken as a reference : a universal hypertext, and no longer the combination of « text - images - links » is what is taken as the vision for the future.

This is only an example and even a non-mathematical one, but the very same ambivalence can be found in the teaching and learning of mathematics. It is obvious that professional mathematics and its practice are deeply influenced by new technologies. But it is less obvious whether these new technologies should, if possible, be used to teach and learn unchanging pieces of knowledge, or if the nature of knowledge itself is undergoing deep change because of the new technology.

3.1.3. Growing Mobility in a Changing Labour and Education Market

With the opening of the European and even the world-wide market to students and workers, schools are confronted with a persistent demand not just to prepare students at an early age for undetermined employment, but also, to provide them with a general, thorough education and with the cultural instruments which will permit them to adapt and update their competencies and knowledge throughout their entire professional life.

We have already remarked earlier that the real consideration of this requirement varies considerably from country to country.

3.2. Some Consequences of the Changes Occurring

With the developments mentioned above, it is not too surprising that in nearly all countries important reforms concerning educational systems are under way. However, the dynamics of change for these educational systems imply difficult challenges :

- It would be desirable to be able to define the basic knowledge and competencies which are necessary for every future citizen coherently and as a result of a social consensus. This consensus is far from being attained even within the same country, if only because the future, by definition, is uncertain. This statement is all the more true for the European Union as a whole.

- It would be desirable to organise the educational system in such a way that all students are able to acquire the basic knowledge and competencies in a conscious, lasting way and in such a way that they can be used not only in a classroom situation, but more importantly, in non-school situations which they will be confronted with later on in life.
- It would be important not to lower the general level of training and education, if only to avoid penalising those students who will later continue their studies after the minimum school-leaving age .

In fact, the idea that the "standards of students' skills is dropping" and that the changes under way contribute to this downgrading is a widely shared conviction among practitioners of mathematics teaching, This is also true for the majority of the countries under study. Observation and analysis may lead to the qualification of this opinion. It may be true that today, more students can learn more material and learn it more effectively than 20 years earlier and that the top 5% of today's generation may have superior competencies when compared to the top 5% some 10 or 20 years ago. Nevertheless, the expectations of society seem to grow even faster than what the students achieve.

> Our society is, as we say, »numerical» but there are not just digitised images ! In fact, numerical information is omnipresent, it is becoming more and more complex and important for each and every citizen. In consequence, it is certainly highly undesirable to have a society in which only a small part of the citizenship is able to understand the world around it. It is not only the training and education of future mathematicians or future professional users of mathematics that is at stake. Obviously, this is an important question concerning democracy itself.

4. Specificity of the Teaching of Mathematics in the Different Countries Concerned by the Study

Besides the aspects already outlined above, which at least, *grosso modo*, all the countries under study have in common, remarkable differences between countries and sometimes even within countries, different regions or parts of the educational systems have to be stated. To give examples, inter-regional differences are significant in Belgium, Germany, Spain and Switzerland…

4.1. The Context of Mathematics Education

It seems that even entire countries remain attached to traditional teaching practices (listening and learning!), while others are very attached to methods such as the « teaching by objectives » approach, and that others are finally in the process of installing a resolutely constructive teaching style.

Here, knowledge comes second, (which does not mean that its importance is totally denied) while procedures and processes are considered the most important. The student is asked to construct his /her knowledge, which at least for some, seems ridiculous and maybe for others is a panacea.

As we already stated, a quasi-universal community of mathematics teachers exists and in all countries, and we can find teachers everywhere who share this constructivistic approach. If we talk about a developed strategy for the entire educational system, we should at least cite the Netherlands, where their experience in this area should be followed with great attention. To a certain extent and with this in mind, Italy, some districts of Switzerland, and some parts of Belgium and the United Kingdom should also be mentioned.

In France, where the constructivistic approach is part of the official approach, it is beginning to be disparaged even if it has not yet begun to be applied. To rely systematically on activities is not enough in order to qualify a constructivistic teaching method

4.2. Place and Importance of Mathematics in the Curriculum

Mathematics is relatively important in most of the curricula. The time at school devoted to mathematics is around 10 to 12% of the total time. However, as this total classroom time varies from country to country, important disparities are beginning to open up. At the age of 16, the average teaching time of mathematics per week varies from 2 hours 15 minutes (3 periods of 45 minutes) to 3 hours 45 minutes, (5 periods of 45 minutes) and for the time being, it is impossible to provide more precise data.

We should additionally mention that the time allocated to mathematics seems to be relatively decreasing under the influence of a rebalancing with other disciplines, particularly in information technology. At the same time, according to an OECD survey, in the majority of countries, mathematics is seen as the most important discipline after the teaching / learning of the first (native) language. The Netherlands is an exception : mathematics comes after information technology and the second language. In the United States, we find an exception : mathematics is designated as the most important discipline. We should add that with rather low results in mathematics (at least if we consider the international studies), the United States is making considerable efforts to improve the situation and it would appear that they are beginning to harvest the fruits of their efforts (see the TIMSS results for population 1).

4.3. Teacher Training

Significant differences can be found between one country and another if only because the basic training of teachers can be carried out either in teacher training universities or in universities open to all, or indeed in other types of establishments with varying degrees of specialisation.

4.4. Major Aims of Mathematics Education

The structure for analysing the question of the objectives is condensed in the table below.

General Abilities	**Mathematical World**	**Application of Math**
Algorithms		
Reasoning, deduction, proof	*Arithmetic*	Modelling
Language (using, creating, communicating ... symbols)	*Variables, equations*	Investigation, (re)search
	Geometry	Approximate calculation
Visual thinking	*Data Analysis*	Computer (-assisted learning.)
Transfer	*Functions (incl. Graphs)*	
Appreciation of mathematics, confidence in using it		Checking of results

This is a way of stressing the fact that we are not only interested in the contents, (the syllabus) but also, and more importantly, in the effects of teaching and learning in terms of acquired competencies.

4.5. Examples of Topics Taught

The manner in which certain questions are treated and evaluated in the countries participating in the study is illustrated in the National Reports. This includes in particular the following themes :

- Quadratic equations
- The theorem of Pythagoras
- Similitude
- Word problems
- Percentages
- Additional topic or approach at choice (age 16) *(innovative topic or approach… complementary relevant information)*.

Concerning outstanding curricular innovations, some countries announce a renewal of interest in the teaching and learning of geometry, and in particular, the geometry of space. A more detailed analysis nevertheless shows that under the heading of geometry, different curricula point to different topics : from the notion of «visual thinking» in the Netherlands (a notion which would not appear to have an equivalent in French) to the calculation of volumes for practical use in other countries. Traditional Euclidean geometry is marginalized in the curricula for nearly all countries. It remains only substantially present in Greece and Italy.

Other countries concentrate innovative efforts on statistics and probability.

In other countries, the stress is directed on the practical use of mathematics in extra-mathematical contexts. Occasionally, there are countries where a growing interest is emerging to integrate into the official curricula elements of the history of mathematics.

5. General remarks on Mathematics and its Teaching & Learning

The developments described above are most pertinent for mathematics in various respects :

- It seems necessary to reflect on the value of mathematical knowledge. For example, mastering the traditional techniques of calculating is less useful than some years before, because hand-held calculators or computers and appropriate software are widely available. On the other hand, the ability to define a calculable programme becomes more important, as well as the ability to check the results. All the same, elementary knowledge in the field of statistics and probability is indispensable for every future citizen.
- It could also be very useful to dwell on the procedures for testing acquired knowledge. The evaluation of knowledge through written work only (which is sometimes only offered in the form of multiple choice) seems to be inappropriate when promoting and evaluating certain types of competencies, for instance, the ability to express oneself correctly, the ability to

express mathematically a non-formalised situation, the ability to work on a project in Groups, and the ability to consult (text)books autonomously, etc ...

- Other reflections of this type inspired particular attention from the participants of the Working Group and are summarised below.

5.1. Conceptions of Teaching and Learning

It is rather difficult to give a realistic image of the way mathematics is taught and learned in the different European countries. Here, we rely on information taken from curricula, textbooks, examination questions and cultural characteristics. These remarks are often subsequently explored in further detail in their respective national reports.

Today, the conceptions of teaching and learning mathematics in Europe are apparent by the reforms that are anticipated or (partly) carried out. Two basic ideas are always at the centre of these changing developments : the traditional concept (when a professor lectures in class what he or she has prepared) and a more innovative approach (called "constructivism", based on the students' activities to construct mathematical notions). All planned or implemented reforms show the same tendency : reduce the lecturing in class and encourage a teaching built more on activities.

But as is often the case in a process destined to reconcile two different conceptions, we were not surprised to note on one hand, a contradiction between the aims of curricula, the content of textbooks and examinations, and on the other hand, the final results obtained (in terms of student knowledge and competencies). Beyond natural contradictions (!), we can distinguish countries which aim at reforming their conceptions (Greece, Hungary, Luxembourg, Portugal, the Russia Federation) from those which are already actually changing their systems (Belgium, France, Italy, Sweden, the United Kingdom...). It is rather difficult to specify all these situations, especially in cases where a federal structure makes these reforms all the more complicated (as in Germany or Switzerland). Certain countries give priority to "mathematics for all" (as in Sweden and the United Kingdom) while others are concerned with the most advanced levels of mathematics (as in France and the Russia Federation).

To give a better description of the difficulties encountered when comparing these various conceptions, we can note for example :

- In certain countries (such as Greece and Luxembourg), textbooks are imposed by an authority responsible for education with sometimes only one authorised textbook, which can induce a high degree of uniformity.

- The exaggerated importance on examinations (like the "baccalauréat" in France) may bring about a kind of «cramming» where the solid construction of mathematical knowledge is perhaps not certain.
- The application of reforms can be facilitated, or, depending on the case, disrupted, by the way in which initial teacher training is organised. In French-speaking Belgium for instance, full or part-time teachers in schools carry out most of the training, and therefore they have a regular contact with the students, a fact that largely facilitates the dissemination of current reforms.

To end this remark, it is of interest to note that various competitions (all kinds of Olympiads, competitions, games ... in France, Hungary or Switzerland ...) have a favourable effect on the relative conceptions of the teaching and learning of mathematics by bringing a feeling of gratification in the resolution of mathematical problems.

In addition to all that was mentioned above, the delicate problem of teacher training must be added. Not only with respect to acquiring new mathematical contents, but more importantly, the acquisition of new teaching methods necessary to face and teach students with extremely varying interests and abilities or even to use new information technology effectively.

5.2. Didactics of Mathematics as a Scientific Discipline

The didactics of mathematics, considered as a scientific discipline in Europe is of great variety : The situation may vary from the near non-existence of didactics of mathematics to its consideration as a «ready-made» scientific discipline (as is the case for example in France). Some European nations are in the process of affirming the didactics of mathematics as a discipline : this is the case of countries which do not yet have university didactic institutions but where elements of this discipline are being developed. For example, this area is currently «undergoing development» in Sweden and is the centre of great interest, in particular, for researchers forming groups who are involved in the research of the didactics of mathematics.

The most remarkable characteristics when understanding the didactics of mathematics as a scientific discipline, are the syllabi and the Chairs of the Didactics of Mathematics already organised within mathematics and/or education departments of universities, scientific journals on the Didactics of Mathematics, national conferences and/or summer university courses and/or the possibility to pursue postgraduate studies (masters / doctorate) of didactics of mathematics.

Let us give some concrete examples of the above situations : Germany has 100 Chairs for the Didactics of Mathematics at post-graduate level. Scientific journals such as, "Recherche en Didactique des Mathématiques" and "Journal für Mathematikdidaktik" are excellent examples of scientific journals on the didactics of mathematics. Italy and Portugal hold important national conferences and seminars, whereas the French summer school on the didactics of mathematics today has a very good international reputation. In addition to that, there are bilateral or even multilateral conferences on the didactics of mathematics in different countries. These characteristics do not mean that these countries have a homogenous conception on the didactics of mathematics. The well established, but patchwork-like of the didactics of mathematics in Germany can be taken as an illustration for this. The didactic of mathematics, reduced to its intuitive and non-research based version on the didactics of mathematics can be developed all too easily.

The description we have given shows that the organisation of a European research community on the didactics of mathematics is highly desirable. The journal "Educational Studies in Mathematics" (an international review published in the Netherlands) shows that communication and co-operation in didactics are possible and enriching. The foundation of the "European Research in Mathematics Education (ERME)" society can be seen as a first step in this direction.

5.3. Heuristics, Proof and Demonstration

In general, it would seem to be important for all citizens to be capable of autonomous and correct reasoning in complex problematical situations and therefore to know how to use hypothetico-deductive reasoning methods. But at the same time it does not seem that the approach of mathematical demonstration in its traditional form could be proposed in a context of education for all. In any case competencies acquired in this context do not seem to be easily transferable to other cultural or work situations. Such reflections, banal or contradictory, deserve some explanations.

Different words are often used as though they were synonyms for "proof" (for example, demonstration, justification, reasoning, explanation). But such expressions have no clear meaning. Indeed, in some countries "justification", "reasoning" and "explanation" are used in a subjective sense, so that they do not imply that what is offered need be even approximately correct - rather they refer to informal utterances reflecting pupils' own thinking. While pupils' contributions are an important part of the learning process, we focus here only on the variety of styles which are open to the "teacher", and which can therefore be incorporated in a formal curriculum. We can therefore distinguish between two styles which are open to the teacher : "proof", and "heuristics".

We will restrict the use of the word "proof" to the designation of a precise and deductive argumentation within some local framework of well-defined explicit or implicit knowledge.

Proof at school level will always be formally incomplete - in the sense that the framework of underlying knowledge is bound to include some aspects which cannot be made explicit at an elementary level (such as "betweenness", and "completeness" in Euclidean geometry). In spite of this restriction, a proof should always respect :

- the careful use of rules of deduction,
- the logical interdependence between definitions, (possibly implicit) axioms, and the hierarchy of previous propositions within the relevant local framework.

We will use the word "heuristics" to indicate mathematical explanations of a more informal, even opportunistic, kind. A heuristic justification lies - at least partly - outside any previously declared local deductive framework, but may draw on any appropriate aspect of pupils' previous mathematical experience. In other words, the heuristical approach:

- will exploit significant and relevant notions, from mathematics, physics or other fields,
- will help pupils of the relevant age to understand a new idea and how to see how it relates to more familiar ideas,
- will call up explanations in a way which are correct "in principle" - in that they can be made fully correct at a higher level (preferably sooner rather than later).

If properly presented, an analogy may hide an isomorphism, and so may satisfy the three preceding criteria. However, in most instances, mere analogy fails one or more of these conditions.

A special case, or illustrative example, may be part of a heuristic justification if it is in some sense paradigmatic or generic ; but often a general result is inferred from one or two special cases without those involved even being aware of the difference between a "paradigmatic case" and a "random instance".

Experimental evidence would usually not qualify as heuristic (as when children tear off corners from paper triangles to demonstrate that the sum of the angles is equal to two right angles) ; but such evidence can sometimes be structured so as to qualify as genuinely heuristic (as when the corners of triangle ABC are folded to meet at the foot L of height AL, since this prefigures the real demonstration based on drawing the parallel to BC through A).

Most European countries declare that "proof" starts around the age of 14 or 15. Moreover, most countries state that this aspect of school mathematics arises mainly in the context of formal geometry. However, when pressed, most such countries acknowledge that significant instances of deduction arise elsewhere (and earlier), but in the context of less imposing (even implicit) local systems. For example,

- in ruler and compass constructions,
- in applications of the fact that "if a, b are real numbers with $a.b = 0$, then $a=0$ or $b=0$",
- in the derivation and extension of the index laws,
- in work involving prime numbers and prime factors ; and elsewhere.

In the context of formal proof – or demonstration - in geometry, most European countries report that, while pupils may achieve a measure of fluency in reproducing standard proofs and in solving related problems, the reasons for the underlying ritual are not generally understood by the majority of those pupils who are officially required to engage in such work on proof.

This observation has to be weighed against the evidence that many pupils find school mathematics appealing precisely because of the logical dependence between different aspects - which means that as one progresses knowledge can be re-organised in a way that continues to reflect the essential simplicity and interconnectedness of the material, without requiring extensive memorisation. This suggests that, even if the global axiomatic ritual is appreciated by relatively few pupils of 16 years or less, the local emphasis on proof and on the logical connections between different parts of the subject may play an important role at secondary level.

This raises the question of whether the latter benefit can be achieved without burdening pupils with an axiomatic ritual, which will be appreciated, by relatively few pupils. The answer is unclear. Here we note only that countries which have concluded that the ritual no longer has a place in secondary mathematics, and who have abandoned the ritual altogether (teaching neither Euclidean geometry nor any pedagogical substitute), have discovered that they appear to have "thrown out the baby with the bathwater". Thus even if the underlying ritual is poorly understood, it may nevertheless contribute in unsuspected ways to pupils' appreciation of the overall interconnectedness of school mathematics in a way that needs to be taken into account in exploring alternative approaches.

What is clear, at least provisionally, is that

- the level of rigorous reasoning can vary from one part of the curriculum to another,
- the underlying local framework within which reasoning takes place is structured more carefully in some areas than in others,
- even if we ignore those countries that have attempted to re-define school mathematics as an experimental subject, there is a clear and widespread trend to look for ways of replacing many formal proofs by heuristic reasoning. This raises many interesting questions.

All these reflexions do not exhaust the subject and beg a number of other interesting questions. For example :

- What are the potential benefits and dangers of a tendency to prefer heuristic reasoning to formal proof ? What criteria need to be observed in order to reap the benefits and to avoid the dangers ?
- To what extent can this be done without undermining the whole unity of school mathematics ?
- How can one ensure that teachers are aware of the relevant conditions that should apply to heuristic reasoning ?
- How can one ensure that pupils appreciate the difference between experimental evidence, heuristic reasoning, and mathematical proof ?

5.4. Links between Mathematics and Information Technology

Should one link traditional teaching of mathematics to the teaching of information technology? What type of mathematics can facilitate the learning of information technology, and of which information technology? And *vice versa,* what are the information technology environments which are likely to favour the learning of mathematics?

> Here we come to some fundamental questions. The conditions of mathematics teaching are in the course of change without the reasons or effects of these changes being really mastered. But there is little doubt that the appearance of new technologies of calculation is the cause of these upheavals.

In all countries, a strong incentive to trivialise new technology is reported, such as the use of not only (scientific) calculators, but also graphic calculators, computers and the Internet. Internet is clearly not of the same nature as the other tools we mentioned, but students are now finding in addition to data bases, simulations of any kind accompanied with an interactivity such that the difference is not immediately obvious. (Be it only that the users interface generally remains a computer screen).

The place and use of handheld calculators and computers in the curricula varies considerably for the following three respects :

- In terms of the place given for the means of electronic calculation in the classroom and during examinations.
- In terms of existing relations between the teaching and learning of mathematics and information technologies.

- In terms of the integration of handheld calculators and computers, and more generally modern instruments for information and communication in the teaching and learning of mathematics.

5.4.1. The Place of New Technologies in the Classroom and at Examinations

We will not dwell on the question of simple calculators ; their use has become widespread from the time of primary school on and hardly concerns students of 16 years old. Scientific calculators are accepted everywhere in the classroom. At the age of 16, the graphic calculator appears systematically in classrooms, and therefore, is considerably modifying the study of functions.

Other calculators (scientific ones, especially ones with graphic screens, programmable and possibly offering formal calculable functions) also tend to be favoured in educational use. In terms of incentive, this is true for all countries, but not in terms of achievement. Ownership of tools like these is not universally widespread, and can even vary within the same country depending on the institution and also depending on the depth of mathematical education (sections with a strong educational background in mathematics compared to those with a weak educational background in mathematics).

There rarely seems to be any systematic training in the use of these instruments. Consequently, teachers complain about a decline of student competencies in the mental manipulation of numeric data. (Here, we are not talking about traditional mental arithmetic, but instead about the necessary skill to handle calculations, even if a calculator is available).

Attitudes concerning the use of handheld calculators in examinations vary considerably. Some countries continue to ban systematically the use of calculators, while others make them nearly obligatory because of the way in which examination problems are given. To give an example, the French ministry of education recently forbade… the prohibiting of the use of modern calculators in all examinations under its control. In some countries, teachers are required to prepare two approaches for the same problem with two types of solution : one with the calculator, the other without.

One of the difficulties encountered by teachers is the large diversity of material used by students. Therefore, it is difficult to organise a systematic introduction of its use. The use of calculators is left to the students' initiative and not really integrated into the teaching process. Let us state here that we have used the terms of calculator and computer to differentiate the two in the compiling of this report. This is still relevant in certain countries but it is generally clear that the differentiation between the two is rapidly becoming obsolete. The difference that one should now insist on is the autonomy or lack of autonomy of the equipment.

Computer presence in school is very unequal. In certain countries (like France and the Netherlands), official norms have been lowered to 8 to 10 students.

But even in these countries, the existing equipment falls short of the recommendations. Moreover, teachers complain about the lack of appropriate software for the teaching and learning of mathematics. In the Netherlands for instance, the majority of mathematics teachers use the computer in class less than 4 times a year. In France, the current situation should not be too different and should be even less favourable in most other countries.

To give an example : even with the widespread availability of dynamical geometry software (DGS) and computer algebraic systems (CAS), their use seems to be a quasi-anecdote in Germany. There again, the situation seems to be similar, or even less favourable in the majority of the other countries.

5.4.2. The Relationship between the Teaching of Mathematics and that of Information Technology

As far as this aspect is concerned, it is clear that the balance of time in terms of teaching hours is swinging in favour of information technology.

In some cases, teachers are trained and recruited as mathematics and information technology teachers, which may allow a minimum coordination between the two disciplines. In other countries, information technology teachers (or possibly technology teachers) assure a teaching of information technology independent from that of mathematics.

But it is far from clear if information technology should serve as a traditional learning tool or should be integrated into the teaching and learning of mathematics.

Teachers sometimes complain about the poor performance of software for computer-based teaching and learning. Sometimes, better programmes do exist, but cannot be introduced into the classroom because of the lack of computer availability in schools.

In many cases, these aids are used to help students with difficulties catch up and are used in places other than the typical classroom setting (for example in documentation centres), or are used in places at the disposal of students outside school - for instance by parental organisations - or, finally, at home or in institutions which offer courses parallel to those of official institutions, this being for a wealthier clientele.

5.4.3. The Integration of Calculators and Computers in the Teaching of Mathematics

The integration of information technology in the teaching of mathematics is appreciated differently. Here it is the concepts themselves which, while keeping the trace of their condition of learning, find themselves changed from an epistemological point of view.

We have known for a long time that the notion of the variable in information technology is not identical to the notion of the variable in mathematics (a concept of utmost mathematical importance). The use of a spreadsheet in mathematics in order to organise and handle algebraic calculations certainly gives a new meaning to the notion of the variable. Is this to say that this is harmful to the teaching and learning of mathematics? We do not think so, but we consider it important to underline this type of shift in meaning, which can sometimes also explain the reluctance (even if implicit) of teachers to integrate information technology into their teaching.

The same type of remark can be made with respect to dynamic geometry software like Cabri where, for example, a point on a line is not exactly a point in the sense of traditional geometry : it can be attached to the straight line, (making it an element of the straight line in the traditional sense) but it can also be freely placed on the straight line (giving it one degree of liberty and we are, in a sense, into mechanics if one wants to use the traditional expressions).

Finally, the arrival of formal calculation software stored in machines like handheld calculators is in the process of disrupting the formal learning conditions of algebraic calculations. Again, at the age of 16 or a little younger, we come across the problems which occurred and still occur concerning the competition with the machine when first learning how to calculate.

In order to complete at least provisionally this overview of the relationship between mathematics and information technology, it needs to be noted that teachers and specialists of the teaching of mathematics everywhere are studying the real effects of these new practices and of these new tools on learning. Even the existence of highly important and promising research projects in this field, as well as the stress placed on the innovating steps in official documents and at the various congresses, should not mask the still very feeble impact of these tools on the majority of practices.

6. Prospects and conclusions

Proposing Reference Levels in mathematics for 16 year-old students for the whole of Europe implies first of all that the conditions of teaching this subject be known and recognised. This study has enabled significant progress in this respect.

It implies also that training and teaching concepts are shared by the concerned partners, including also their conceptions on the kind of mathematics taught and on the important aspects of their teaching.

The study has enabled significant progress to be made in this second direction. Communication, exchanges, dialogues were able to take place among a network of mathematicians who were all concerned by improving the teaching of their subject while respecting local particularities.

What is proposed in this report is an instrument for communication and dialogue with and between mathematics teachers, with experts in teacher training and with those responsible for teaching systems.

There is still a lot to be done to complete the work of our group and to make it operational, as there is still a lot to do in order to ensure its diffusion at all levels.

In order for mobilisation on these issues to be achieved, at least as far as the field of mathematics is concerned, there is still much to be accomplished.

The embryo of a resource centre on curricula and mathematical evaluation that we have set up deserves continuous expansion and updating as equally does its associated web site.

As important would be the prolongation of the study for 18 year-old students, that is at secondary / post-secondary transition level.

Different reports from the study and other documents on mathematical teaching practices in Europe are available on and can be down-loaded from our web-site :

http://www-math.univ-fcomte.fr/DEPARTEMENT/CTU/IREM/internat.htm

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